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Proceeding Paper

Majorana Mass Term and CAR Algebra of Creation and Annihilation Operators of Dirac and Majorana Spinors †

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Abstract: We have proved that, under the standard charge conjugation approach, the Majorana mass term in QFT must vanish. We have derived formulas for the Majorana spinor field operator without any assumptions about second quantization procedure. The fact that the Majorana mass term vanishes not only in the c-theory, which was known, but also in the q-theory (the theory of second quantization), requires a revision of ideas about the generation of neutrino mass using the seesaw mechanism.

Keywords: Majorana mass; Majorana spinors; neutrino mass; seesaw; see-saw; second quantization

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1. Introduction

Solutions of the Dirac equation, which are invariant under the charge conjugation operation, were introduced by Majorana in 1937 [1]. These solutions are called Majorana spinors, or, which is the same, Majorana fermions.

2. The Charge Conjugation and Majorana Spinors

We will consider space-time with the metric tensor $g^{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$. In the case of the considered signature, the Dirac matrices have the form

$$\gamma_D^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix};\ \gamma_D^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix},\ k = 1,2,3;\ \gamma_D^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{1}$$

where σ_k are Pauli matrices and I is the identity 2×2 matrix. Further, we will indicate from below the index D for the Dirac representation of the gamma matrices, W for the Weyl (chiral) representation, and M for the Majorana one.

$$\gamma_W^0 = \gamma_D^5,\ \gamma_W^k = \gamma_D^k, k = 1,2,3;\quad \gamma_W^5 = -\gamma_D^0. \tag{2}$$

In the Dirac and Weyl representations, the gamma matrices $\gamma_W^k = \gamma_D^k$, $k = 1, 2, 3$, are the same, so we will not indicate the indices D and W for them.

Majorana [1] proposed to consider solutions of the Dirac equation

$$\gamma^\mu (i\partial_\mu - qA_\mu)\,\psi = m\psi\,, \tag{3}$$

in a representation in which all gamma matrices are imaginary. In this case, the charge conjugation condition $\psi = \psi^c$ coincides with the condition that the spinor is equal to the complex conjugated

$$\psi = \psi^* . \quad (4)$$

For such a solution, the requirement $q = 0$ in (3) must be satisfied.

The charge conjugation operator in the Dirac and Weyl representations is

$$(\cdot)^c = \eta_1 \gamma^2 (\cdot)^* , \quad (5)$$

where η_1 is complex phase factor, $|\eta_1| = 1$.

Any solution of the Dirac equation can be represented as the sum of the real and imaginary parts (in the sense of complex conjugation)

$$\begin{aligned} &\psi = \Psi_1 + i\,\Psi_2 , \\ &\psi_M = \Psi_1 = (\Psi_1)^* = (\Psi_1)^c = \frac{1}{2}(\psi + \psi^c); \\ &\Psi_2 = (\Psi_2)^* = (\Psi_2)^c = \frac{i}{2}(\psi^c - \psi). \end{aligned} \quad (6)$$

All charges of these solutions must be equal to zero. Majorana concluded that such a completely neutral fermion must coincide with its antiparticle.

Let us define

$$\psi_+ = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \quad (7)$$

Then in the Dirac or Weyl representation we have

$$\psi_{+M} = \frac{1}{2}(\psi_+ + \eta_1 \gamma^2 \psi_+^*) = \frac{1}{2}\begin{pmatrix} \phi \\ -\eta_1 \sigma_2 \phi^* \end{pmatrix}. \quad (8)$$

Let

$$\chi_1 = i\eta_1 \phi'^*_2 , \quad \chi_2 = -i\eta_1 \phi'^*_1 , \quad (9)$$

where ϕ'_1 and ϕ'_2 are components of some spinor. Consequently,

$$\psi_- = \begin{pmatrix} 0 \\ \chi \end{pmatrix}, \quad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \quad (10)$$

Then

$$\psi_{-M} = \frac{1}{2}(\psi_- + \eta_1 \gamma^2 \psi_-^*) = \frac{1}{2}\begin{pmatrix} \eta_1 \sigma_2 \chi^* \\ \chi \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \phi' \\ -\eta_1 \sigma_2 \phi'^* \end{pmatrix}. \quad (11)$$

We have obtained formula (11) for ψ_{-M}, similar to formula (8) for ψ_{+M}. Thus, the Majorana spinor ψ_M can be constructed either from ψ_+, or from ψ_-, or from their sum. These constructions are equivalent. In the Dirac representation, ψ_- are the positive-frequency components of the spinor, and ψ_- are negative-frequency. In the Weyl representation, ψ_- are left chiral components, and ψ_- are right chiral ones.

3. The Problem of the Vanishing of the Majorana Mass Term

The mass term L_m in the Lagrangian corresponding to the Dirac equation can be written as

$$L_m = -m\overline{\psi}\psi = -m\psi^+\gamma^0\psi = -(m\psi^+\gamma^0\psi)^+ \tag{12}$$

where $\overline{\psi} = (\gamma^0\psi)^+$ is the Dirac conjugated spinor.

For the Majorana spinor in the Dirac representation (8), we get for it

$$L_{mD} = -m\left((\phi_1^{\ +}\phi_1 + \phi_2^{\ +}\phi_2) - (\phi_1^{\ +}\phi_1 + \phi_2^{\ +}\phi_2)^*\right). \tag{13}$$

In the Weyl representation

$$L_{mW} = -m\left((\phi_1'^T\,\phi_2 - \phi_2^T\phi_1') - ((\phi_1'^T\,\phi_2 - \phi_2^T\phi_1'))^*\right), \tag{14}$$

where $\phi_1' = i\,\eta_1^{\ -1}\phi_1$, and we see the absence of dependence on η_1 in (13) and (14).

If we do not use second quantization, that is, in the so-called c-theory, ϕ_1 and ϕ_2 are complex numbers. That is why $\phi_1^{\ +} = \phi_1^{\ *}$, $\phi_2^{\ +} = \phi_2^{\ *}$ in (13) and $\phi_1'^T = \phi_1'$, $\phi_2^T = \phi_2$ in (14), and we have zero Majorana mass term

$$L_m = 0. \tag{15}$$

It is known that the Majorana mass term vanishes in the c-theory [2,3]. In [2] it is assumed that this problem can be solved in the q-theory (second quantization theory). Let us show that this is not the case.

The mass term is Lorentz invariant, so it suffices to consider its value at zero spatial momentum. We will use the Majorana-Dirac representation of the gamma matrices. The Dirac equation at zero spatial momentum

$$\gamma^0_{MD} i\partial_0\ \psi_M = m\psi_M\,, \tag{16}$$

$$\gamma^0_{MD} = -\gamma^2\gamma^0_D = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \tag{17}$$

$$\psi_M = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{18}$$

Therefore, Equation (16) can be rewritten as

$$\begin{aligned} \ddot{\psi}_1 &= -m^2\psi_1, \\ \ddot{\psi}_2 &= -m^2\psi_2, \\ \psi_3 &= \frac{1}{m}\partial_0\psi_2, \\ \psi_4 &= -\frac{1}{m}\partial_0\psi_1. \end{aligned} \tag{19}$$

Solutions of (19) are obvious

$$\begin{aligned} \psi_1(t) &= \ \ a_1\ \cos(mt+\varphi_1), \\ \psi_2(t) &= \ \ a_2\ \cos(mt+\varphi_2), \\ \psi_3(t) &= -a_2\ \sin(mt+\varphi_2), \\ \psi_4(t) &= \ \ a_1\ \sin(mt+\varphi_1), \end{aligned} \tag{20}$$

where $t = x^0$ is time, $a_1 = \psi_1(0)$, $a_2 = \psi_2(0)$, and φ_1, φ_2 are phases depending on the initial conditions. In the theory of second quantization, a_1 and a_2 are operators of annihilation and creation of the Majorana spinors (with some normalization factor).

The mass term (12) in this case is

$$L_m = -m\psi_M{}^{+}\gamma^0_{MD}\psi_M = -mi\,(-a_1{}^{+}a_1\cos(mt+\varphi_1)\sin(mt+\varphi_1) - a_2{}^{+}a_2\cos(mt+\varphi_2)\sin(mt+\varphi_2) + a_2{}^{+}a_2\cos(mt+\varphi_2)\sin(mt+\varphi_2) + a_1{}^{+}a_1\sin(mt+\varphi_1)\cos(mt+\varphi_1)) = 0. \quad (21)$$

Thus, the Majorana term of the Lagrangian is equal to zero both in c-theory and in the theory of second quantization (q-theory). Thus, the existence of Majorana spinors without an admixture of Dirac spinors is impossible. Such Majorana spinors arise in the seesaw mechanism due to the diagonalization of the symmetric mass matrix. However, we have not refuted the possibility of the existence of Majorana-Dirac fermions with a mixture of Majorana and Dirac components and an asymmetric mass matrix. Therefore, the seesaw mechanism can probably be modified by replacing the symmetrical mass matrix with an asymmetric one that cannot be fully diagonalized.

4. The Field Operator and Hamiltonian of the Majorana Spinor

Let us pass to the Dirac representation using the transformation inverse to (17). In this case, spinor (18) will be multiplied by matrix

$$(V_1)^{-1} = \left(\eta_2\,\frac{1-\gamma^2}{\sqrt{2}}\right)^{-1}, \quad (22)$$

where $\eta_2 = \eta_1^{-1/2}$. Therefore, denoting by ψ_{DM} the field operator of the Majorana spinor in the Dirac representation, we obtain from (18), (20) and (22)

$$\psi_{DM} = \eta_2^{-1}\,\frac{1+\gamma_D^2}{\sqrt{2}}\psi_M = \frac{\eta_2^{-1}}{\sqrt{2}}\begin{pmatrix} a_1\cos(mt+\varphi_1) - ia_1\sin(mt+\varphi_1) \\ a_2\cos(mt+\varphi_2) - ia_2\sin(mt+\varphi_2) \\ -a_2\sin(mt+\varphi_2) + ia_2\cos(mt+\varphi_2) \\ a_1\sin(mt+\varphi_1) - ia_1\cos(mt+\varphi_1) \end{pmatrix} = \frac{\eta_2^{-1}}{\sqrt{2}}\begin{pmatrix} a_1\exp(-i(mt+\varphi_1)) \\ a_2\exp(-i(mt+\varphi_2)) \\ ia_2\exp(i(mt+\varphi_2)) \\ -ia_1\exp(i(mt+\varphi_1)) \end{pmatrix} \quad (23)$$

Formula (23) was derived for the case $p = 0$, and it does not take into account the normalization of the spinor. It can be easily generalized to the case of arbitrary p and with normalization taken into account. To do this, it suffices to multiply (23) by the Lorentz boost matrix and the normalization factor as well as replace mt by $p_\mu x^\mu$

$$\psi_{DM'} = \eta_2^{-1}\,n(p)\,\big(a_1(p)u_1\exp(-i(p_\mu x^\mu+\varphi_1)) + a_2(p)u_2\exp(-i(p_\mu x^\mu+\varphi_2)) + ia_2(p)v_1\exp(i(p_\mu x^\mu+\varphi_2)) - ia_1(p)v_2\exp(i(p_\mu x^\mu+\varphi_1))\big), \quad (24)$$

where u_1, u_2, v_1, v_2 are the usual spinor columns in the Dirac representation and $n(p)$ is the normalization factor. For Dirac spinor [4] $n(p) = \dfrac{1}{(2\pi)^{3/2}}\sqrt{\dfrac{m}{E}}$. Since the Dirac spinor is a superposition of two Majorana spinors with the same mass [2,5–7], it is necessary to retain this normalization for Majorana spinors. Moreover, these field operators must be elements of the same CAR algebra. By construction, operators a_1 and a_2 in (20) are real, that is, $a_1^* = a_1$ and $a_2^* = a_2$.

Now consider the Hamiltonian of the Majorana spinor at zero momentum. The Lagrangian L corresponding to the Dirac Equation (3) for $q = 0$ can be written as

$$L = \frac{1}{2}\overline{\psi}_M \gamma^\mu i\partial_\mu \psi_M + \frac{1}{2}(\overline{\psi}_M \gamma^\mu i\partial_\mu \psi_M)^+ - m\overline{\psi}_M \psi_M . \quad (25)$$

Corresponding Hamiltonian is

$$H = \frac{\partial L}{\partial \dot{\psi}_M}\dot{\psi}_M - L = \frac{\partial L}{\partial \dot{\psi}_M}\dot{\psi}_M = \overline{\psi}_M \gamma^0 i\partial_0 \psi_M = \psi_M^+ i\partial_0 \psi_M . \quad (26)$$

Substituting in (26) values of ψ_M from (20) corresponding to $p = 0$, we obtain

$$H = 0 . \quad (27)$$

This result follows from Equations (16) and (21).

For Dirac spinors, the same reasoning gives for $p = 0$ the Hamiltonian corresponding to the mass term of the Lagrangian [4].

5. Conclusions

We have proved that, under the standard charge conjugation approach, the Majorana mass term must vanish not only in the so called c-theory, which was known, but also in the q-theory (the theory of second quantization). In this case, the possible influence of the phase factor during charge conjugation is taken into account. It turned out that it does not affect the result.

We have derived formulas for Majorana spinor field operators without any assumptions about second quantization procedure. We have proved that the Hamiltonian of the Majorana spinor at zero momentum is zero.

The fact that the Majorana mass term and the Hamiltonian of the Majorana spinor vanish at zero momentum requires a revision of ideas about the generation of neutrino mass using the seesaw mechanism.

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