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Proceeding Paper

Physical Picture of Electron Spin †

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Abstract: Pauli established the standard view that the spin of the electron was a completely abstract, non-classical angular momentum, that could not be thought of as the rotation of anything. Here we give a pedagogical presentation of old work by Belifante (1939) recently updated by Ohanian (1986) which shows that contrary to Pauli's edict, the spin of the electron can be viewed as the rotational angular momentum in the wave field of the electron.

Keywords: spin angular momentum; electron structure; Dirac field

1. Introduction

The standard view of the spin of the electron is that it is an internal angular momentum which cannot be pictured as a tiny rotating ball or in fact as the rotation of anything [1]. If one were to picture the spin of the electron as coming from the rotation of a spherical shell of radius xxx, mass, xxx and having an angular momentum of hbar/2 then one would find that points on the surface of this spherical shell would need to move a ~100 the speed of light, which is impossible. Based on these types of arguments Pauli declared that no two electrons have the same quantum states.

However, quite early in the formulation of quantum theory, Belinfante [2] and Gordon [3] argued that one could interpret the spin and a rotating angular momentum coming from a rotating energy-momentum in the Dirac field the describes the electron.

Here, we review the arguments of Belinfante and Gordon, as well as more recent work by Ohanian which shows in details how the spin of the electron can be seen as a rotation of energy-momentum in the Dirac wave field. This shows that electron spin is exactly of the same character as any other angular momentum, rather than some mysterious quantum property. We start by showing how this view point can be applied to the electromagnetic field to get the spin of the photon and then we move to the Dirac field to obtain the spin of the electron.

2. Spin of the Electromagnetic Field

In this section we obtain the spin of the photon using the same method we will use in the next section to obtain the spin of the electron. The reason for this is that this shows the connection between and commonality of the spin coming from Maxwell's equations and the Dirac equation.

The momentum density carried by the electromagnetic field is $\vec{G} = \frac{\vec{E} \times \vec{B}}{\mu_0 c^2}$. From this one can obtain the angular momentum density as $\vec{\mathcal{J}} = \frac{\vec{x} \times (\vec{E} \times \vec{B})}{\mu_0 c^2}$

For example, it depicts the time averaged transverse energy flow of a circularly polarized wave is moving in the direction of z. The wave exhibits cylindrical symmetry in the z axis and a finite extent in x and y directions. The wave also has a translational energy in z-direction but the net energy flow is helical.

Expressing the net angular momentum as a sum of two terms:

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$$\vec{G} = \frac{\vec{E} \times \vec{B}}{\mu_0 c^2} = \frac{\vec{E} \times (\nabla \times \vec{A})}{\mu_0 c^2} \tag{1}$$

Using the triple product vector formula; we have

$$\vec{G} = \frac{[E^n \nabla A^n - (\vec{E}.\nabla)\vec{A}]}{\mu_0 c^2}$$

Correspondingly, the angular momentum is a sum of two terms

$$\vec{J} = \frac{1}{\mu_0 c^2} \int x \times (\vec{E}^n \nabla \vec{A}^n) d^3x + \frac{1}{\mu_0 c^2} \int x \times [-(\vec{E}.\nabla)\vec{A}] d^3x \tag{2}$$

Using an integration by parts, with $\nabla . \vec{E} = 0$, then it becomes as,

$$\vec{J} = \frac{1}{\mu_0 c^2} \int x \times (\vec{E}^n \nabla \vec{A}^n) d^3x + \frac{1}{\mu_0 c^2} \int (\vec{E} \times \vec{A})\, d^3x \tag{3}$$

The first term in above equation represents the orbital angular momentum and the second term represents the spin.

To justify this interpretation, consider a circularly polarized plane wave with vector potential.

$$\vec{A} = (\hat{x} \pm i\hat{y}) \left(\frac{iE_0}{\omega}\right) e^{i\omega(t - \frac{z}{c})}$$

The time-average values of the integrals from Equation (3) are

$$\vec{L} = \frac{1}{2\mu_0 c^2} \int Re\, (x \times (E^n \nabla A^{*n}) d^3x)$$

$(E^x \nabla A^{*x}) = \frac{{E_0}^2}{c}\hat{z}$, What are you doing here?

$(E^y \nabla A^{*y}) = \frac{{E_0}^2}{c}\hat{z}$ What are you doing here? Explain this step

Therefore, $E^n \nabla A^{*n} = \frac{2{E_0}^2}{c}\hat{z}$

Substitute in above equation, we get,

$$\vec{L} = \frac{1}{\mu_0 c^3} \int \vec{x} \times (\hat{z}{E_0}^2) d^3x \tag{4}$$

Now,

$$S = \frac{1}{2\mu_0 c^2} \int Re\, (E \times A^*) d^3x \tag{5}$$

$\mathrm{E} = (\hat{x} \pm i\hat{y})\, E_0 e^{i\omega\left(t - \frac{z}{c}\right)}(e^{\alpha})$, Assume e^{α} as exponential term and alpha is the vector.

Now, consider $E \times A^*$; we get $E \times A^* = \pm \frac{2{E_0}^2}{\omega}\hat{z}$

Therefore,

$$\mathrm{S} = \pm \frac{{E_0}^2}{\mu_0 c^2 \omega} \int d^3x\, \hat{z}$$

The first of these expressions is polarization independent, and is exactly what we expect for the plane wave's orbital angular momentum. Because, the second expression is not affected by the polarization. We must identify it as the spin. But, the individual integrals in equation are not gauge invariant.

The energy density in the wave equation is

$$U = \frac{1}{2\mu_0 c^2} \int Re\, (E.E^*) d^3x \tag{6}$$

Here, $\mathrm{E} = (\hat{x} \pm i\hat{y})\, E_0 e^{i\omega\left(t - \frac{z}{c}\right)}$ and $E^* = (\hat{x} \mp i\hat{y})\, E_0 e^{-i\omega\left(t - \frac{z}{c}\right)}$

Now, consider the real part of the energy density in Equation (6), then we get

$$E.E^{*} = E_0{}^2 + E_0{}^2 \text{ i.e., } Re(\mathrm{E}.E^{*}) = 2E_0{}^2$$

Then, the energy density equation becomes as (7)

$$U = \frac{1}{\mu_0 c^2}\int E_0^2\, d^3x \tag{7}$$

From spin and energy density equations, we have $\frac{S_z}{U} = \pm\frac{1}{\omega}$.

If we normalize the wave, then the energy is one quantum U = $\hbar\omega$, then the spin will be $S_z = \pm\hbar$.

3. Dirac Field Angular Momentum

I am done with the similar calculations done in Section 1 for the electromagnetic field, but now for the Dirac field [4]. First the momentum density of the Dirac field is given by

$$\vec{G} = \frac{\hbar}{4i}(\psi^{\dagger}\nabla\psi - \psi^{\dagger}\vec{\alpha}\partial_t\psi) + hc$$

where *hc* stands for Hermitian conjugate, ψ is the Dirac spinor field, and $\vec{\alpha}$ are Dirac matrices may be giving a cite to the standard form of the alpha. The time derivative in above equation can be eliminated by means of Dirac equation $\frac{1}{c}\frac{\partial\psi}{\partial t} = \left(-\vec{\alpha}.\nabla + \frac{mc^2}{i\hbar}\beta\right)\psi$.

It gives,

$$\vec{G} = \left(\frac{\hbar}{4i}\right)(\psi^{\dagger}\nabla\psi + \psi^{\dagger}\vec{\alpha}(\vec{\alpha}.\nabla)\psi) + hc$$

Then the commutation relations for

$$\alpha_k, \vec{G} = \left(\frac{\hbar}{2i}\right)(\psi^{\dagger}\nabla\psi - (\nabla\psi^{\dagger})\psi) + \frac{\hbar}{4}\nabla\times(\psi^{\dagger}\sigma\psi)$$

where $\sigma_1 = -i\alpha_2\alpha_3$, $\sigma_2 = -i\alpha_3\alpha_1$, $\sigma_3 = -i\alpha_1\alpha_2$

The first term in above equation is the translational motion of the electron, whereas the second term is the circulating flow of energy.

For example, consider the Gaussian packet,

$$\psi = (\pi d^2)^{-\frac{3}{4}} e^{-\left(\frac{1}{2}\right)\frac{r^2}{d}}\,\omega^{|}(0);$$

It defines an electron spin with zero expectation value of the momentum.

From above equation, the first term is zero and the second term is,

$$\boldsymbol{\vec{G}} = \frac{\hbar}{4}\left(\frac{1}{\pi d^2}\right)^{3/2}\frac{e^{-\frac{r^2}{d^2}}}{d^2}(-2y\,\hat{x} + 2x\hat{y})$$

In case of an electromagnetic wave, the angular momentum rises due to a circulating flow of energy. Then, the angular momentum is the spin of an electron. Therefore, the net angular momentum is,

$$\vec{J} = \int\frac{\hbar}{2i}x\times[\psi^{\dagger}\nabla\psi - (\nabla\psi^{\dagger})\psi]d^3x + \int\frac{\hbar}{4}x\times[\nabla\times(\psi^{\dagger}\sigma)\psi]d^3x \tag{8}$$

Using the triple cross product in the second term can be expanded into two dot products and then integrate both of these in by parts. Then it gives,

$$\vec{J} = \frac{\hbar}{2i}\int x\times[\psi^{\dagger}\nabla\psi - (\nabla\psi^{\dagger})\psi]d^3x + \frac{\hbar}{2}\int\psi^{\dagger}\vec{\sigma}\,\psi\, d^3x \tag{9}$$

Here, the spin is the second term and the first term is the orbital angular momentum.

From the spin in above equation, the expectation value of the quantum mechanical operator $\vec{\sigma}$, the operator representing the spin must be

$$S_{op} = \frac{\hbar}{2}\vec{\sigma}$$

It yields the value $\pm\frac{\hbar}{2}$ for the integral S_z.

4. Summary and Conclusions

As stated in previous sections, spin is fundamentally a quantum mechanical feature. The type of wave is less important in these calculations than the fact that spin is ultimately a wave feature. The spin of a classical wave is a continuous macroscopic quantity, whereas quantum spin is quantized and represented by a quantum mechanical operator. This is the fundamental distinction between the spins of the two types of waves. Because the spin of a quantum mechanical particle has a fixed magnitude, reaching the classical limit is impossible.

We first calculated the spin of the photon starting with the Maxwell's equations and the spin of the electron in the next section.

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