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# Transformed Log Burr III Distribution: Structural Features and Application to Milk Production †

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**Abstract:** The Burr III distribution is extended in this work as a substitute for the numerous Burr III distributions. A new distribution is developed by applying the log transformation technique to define the transformed log Burr III distribution. Moments and quantile function are the structural features established in this study. The model parameters are derived using the maximum likelihood technique. The applicability of the new distribution was assessed using real-world data on the transformed total milk production in the first birth of 107 cows of the SINDI race. The results showed that the proposed distribution might be used as the optimal distribution for this data set.

**Keywords:** burr iii distribution; failure rate; moments; quantile function; transformed log-burr iii distribution

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## 1. Introduction

Burr (1942) developed a flexible family of probability distributions that can be derived from a single differential equation [1]. Two members of this family, the Burr types III and XII, have been introduced by [2]. These distributions are crucial and frequently employed for modeling many real-life phenomena in diverse areas of application, including ecology, agriculture, finance, survival analysis, forestry, medical sciences, reliability quality control, mechanical factors, life distributions, risk analysis, weather forecasting, consumer prices, and more [3].

There are both theoretical and practical reasons for us to propose this distribution. Theoretically speaking, the tail of a distribution is closely related to the distributions of extreme values [4]. In practice, one prefers a distribution that is flexible. The flexibility of Burr XII distribution has been studied in [5]. Compared with the Burr XII distribution, the Burr III distribution is more flexible in the sense that it covers a larger area in the skewness–kurtosis space. The cumulative distribution function (CDF) of the Burr III distribution is expressed as:

$$F_Y(y;\beta,\lambda)=\left(1+y^{-\beta}\right)^{-\lambda};\quad \beta,\lambda>0;\ y>0 \tag{1}$$

where $\beta,\lambda>0$ are two shape parameters. The corresponding probability density function (PDF) is expressed as:

$$f_Y(y;\beta,\lambda)=\beta\lambda y^{-\beta-1}\left(1+y^{-\beta}\right)^{-\lambda-1}, \tag{2}$$

Recently, many extensions of the Burr III distribution have been generated to provide more flexibility in modeling real life data sets from a variety of applications. Some notable among them are the beta Burr III distribution [6], applied to survival data, and the extended three-parameter Burr III distribution [4].

The extension of well-known distributions for modeling real data via generalized classes of distributions has received considerable attention during the last decade. In particular, the recent distributions proposed using the T-X approach include the exponentiated odd Lindley-*X* family by [7], the Maxwell-Weibull distribution by [8], the Maxwell-Lomax distribution by [9], the Maxwell-exponential distribution by [10], the odd beta prime-G family by [11], the odd beta prime-logistic distribution by [12], the odd beta prime Fréchet distribution by [13], the log-Topp-Leone distribution by [14], and more others.

The aim of this research is to apply the logarithmic transformation to the Burr III distribution to create a new effective and flexible distribution referred to as the log-Burr III distribution. The log-Burr III distribution is proposed by the logarithmic transformation of the famous Burr III distribution. To the best of our knowledge, this is the first attempt to study the log-Burr III distribution in the literature.

The log-Burr III distribution is derived from the Burr III distribution in a manner similar to how the log-normal distribution is derived from the normal distribution. The resultant distribution has a long tail since the logarithmic transformation reduces a large observation to a small value. This study investigates whether this distribution is appropriate for modeling real data. We verify in the application section that the log-Burr III model is a better model for symmetrical and left-skewed data sets and can serve as an alternative to various extended versions of the Burr III distribution in many practical situations.

The following summarizes the main motivations for proposing the log-Burr III distribution.

i. The log-Burr III distribution provides better fit than the traditional Burr III distribution.
ii. The Burr III distribution offers symmetrical, and left-skewed densities with an upside-down bath-tub and decreasing failure rates.
iii. The Burr III distribution was applied to fit a long-tailed real data, and it provides superior fits than the other competing distributions.

The rest of this paper is outlined as follows: Section 2 presents the log-Burr III distribution alongside its PDF and hazard rate function plots. Section 3 investigates some of its basic features. Section 4 discusses its parameter estimation method. Section 5 demonstrates its usefulness and effectiveness by analyzing real data relating to Milk Production. Section 6 provides the concluding remarks.

## 2. Developing Log Burr III Distribution

In this section, we developed a novel continuous probability distribution to serve as an alternative to the Burr III distribution using a transforming approach. The novel distribution is developed by transforming $x^{\alpha} = \log(y)$ into the Burr III model to study the Log Burr III (LBIII) distribution, where $\alpha > 0$ is an exponent parameter. The PDF of the proposed distribution can be obtained by considering

$$f_X(x;\alpha,\beta,\lambda) = f_Y(y;\beta,\lambda) \times \left|\frac{dy}{dx}\right| \tag{3}$$

In this regard, $f_Y(y;\beta,\lambda)$ is defined in (2) and $\frac{dy}{dx} = \alpha x^{\alpha-1} e^{x^{\alpha}}$ is the derivative of the transformed approach considered in this study. Therefore, the proposed LBIII distribution can be expressed as

$$f_X(x;\alpha,\beta,\lambda) = \alpha\beta\lambda x^{\alpha-1}e^{-\beta x^{\alpha}}\left(1+e^{-\beta x^{\alpha}}\right)^{-\lambda-1}; \qquad \alpha,\beta,\lambda>0;\ x\in(-\infty,\infty) \quad (4)$$

where $\alpha$ is an exponent parameter and $\beta,\lambda$ are two shape parameters. Henceforth, the CDF of the LBIII distribution can be derived by differentiating (4) with respect to $x$ as

$$F_X(x;\alpha,\beta,\lambda) = \left(1+e^{-\beta t^{\alpha}}\right)^{-\lambda}; \qquad \alpha,\beta,\lambda>0;\ x\in(-\infty,\infty) \quad (5)$$

*2.1. Model Validity Check*

To determine whether the suggested LBIII distribution is a valid statistical distribution, the PDF in (4) must satisfy the following condition:

$$\int_{-\infty}^{\infty} f_X(x;\alpha,\beta,\lambda)dx = 1 \quad (6)$$

To demonstrate this, consider substituting (4) as

$$\int_{-\infty}^{\infty} f_X(x;\alpha,\beta,\lambda)dx = \alpha\beta\lambda\int_{-\infty}^{\infty} x^{\alpha-1}e^{-\beta x^{\alpha}}\left(1+e^{-\beta x^{\alpha}}\right)^{-\lambda-1}dx.$$

$$= \lambda\int_{0}^{\infty}(1+m)^{-\lambda-1}dm, \quad (7)$$

since,

$$m = e^{-\beta x^{\alpha}}, \ \text{and}\ dx = -\frac{dm}{\alpha\beta x^{\alpha-1}e^{-\beta x^{\alpha}}}. \quad (8)$$

Also, letting

$$\frac{1}{w} = (1+m),\ \ w = \frac{1}{1+m},\ \Rightarrow\ dm = -\frac{dw}{w^2} \quad (9)$$

Putting (9) into (8) we can receive

$$\int_{-\infty}^{\infty} f_X(x;\alpha,\beta,\lambda)dx = \lambda\int_{0}^{1} w^{\lambda-1}dw = 1. \quad (10)$$

The proposed LBIII model is a legitimate statistical distribution, as demonstrated by Equation (11).

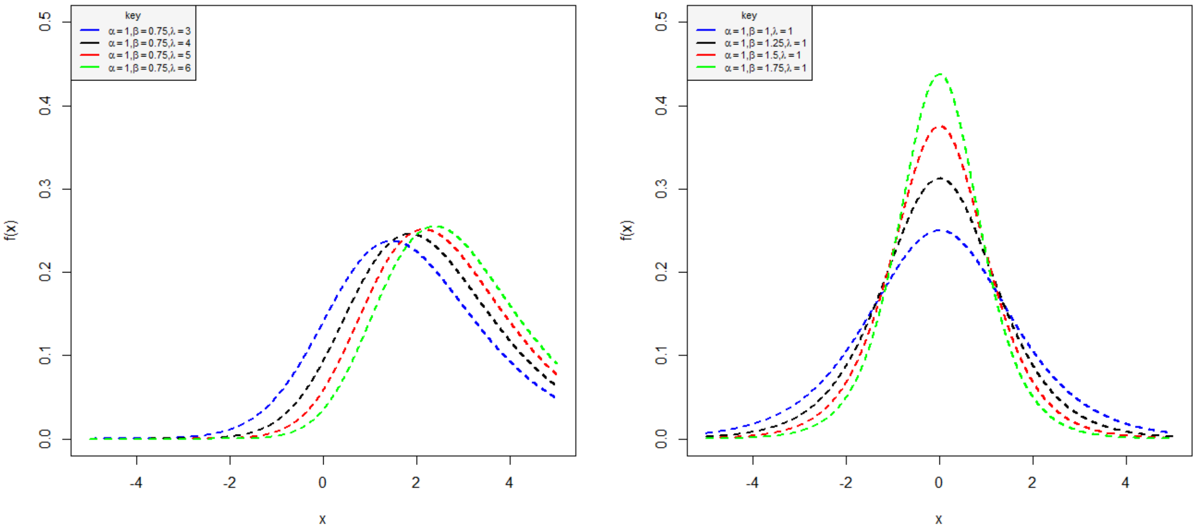
*2.2. Failure Rate*

The failure rate of the proposed LBIII distribution can be determined using (4) and (5) as

$$h(x;\alpha,\beta,\lambda) = \frac{\alpha\beta\lambda x^{\alpha-1}e^{-\beta x^{\alpha}}\left(1+e^{-\beta x^{\alpha}}\right)^{-\lambda-1}}{1-\left(1+e^{-\beta t^{\alpha}}\right)^{-\lambda}}; \qquad \alpha,\beta,\lambda>0;\ x\in(-\infty,\infty) \quad (11)$$

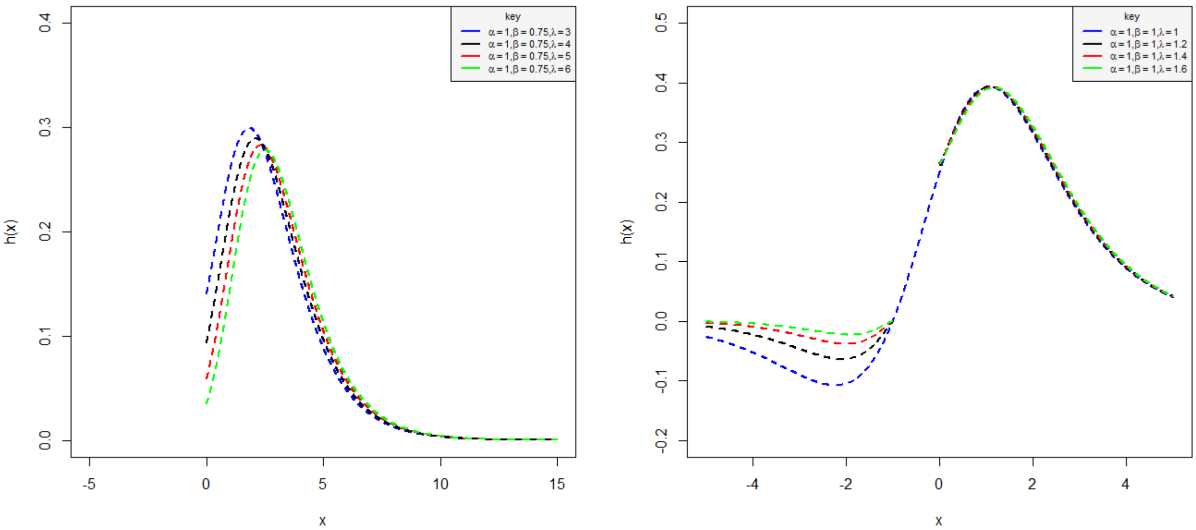
The probability plots for the PDF and failure rate of the LBIII distribution are presented in Figures 1 and 2 respectively.

As shown in Figure 1, the LBIII distribution can be left-skewed and symmetric, as shown in (a) and (b). Similarly, the failure rate of the distribution could have both decreasing and upside-down bath-tub, as shown in Figure 2a,b.




(**a**) (**b**)

**Figure 1.** PDF plots of the LBIII distribution for various parameter values.




(**a**) (**b**)

**Figure 2.** Failure rate plots of the LBIII distribution for various parameter values.

### *2.3. Mixture Representations*

The PDF of the LBIII distribution can also be presented as a mixture representation study in this section using the following procedure:

Let us consider the generalized binomial expansion when $c > 0$ as

$$(1+\psi)^{-c} = \sum_{i=0}^{\infty}(-1)^{i}\binom{c+i-1}{i}\psi^{i} \tag{12}$$

Putting (12) into the PDF from (4) yields

$$f_X(x;\alpha,\beta,\lambda) == \alpha\beta\lambda x^{\alpha-1}\sum_{i=0}^{\infty}(-1)^i\binom{\lambda+i}{i}e^{-(i+1)\beta x^{\alpha}} \tag{13}$$

which is the PDF of the LBIII distribution expressed as a mixture representation.

## 3. Structural Features of the LBIII Distribution

This section describes some features of the LBIII distribution, such as moments and the quantile function.

### 3.1. Moments

Suppose $X$ is a random variable that follows LBIII distribution, then the moments of the random variable $X$ is defined as

$$E\left(x^r\right) = \int_{-\infty}^{\infty} x^r f_X(x;\alpha,\beta,\lambda)dx \tag{14}$$

where $f_X(x;\alpha,\beta,\lambda)$ is defined in (13). Substituting (13) into (14) it becomes

$$E\left(x^r\right) = 2\alpha\beta\lambda\sum_{i=0}^{\infty}(-1)^i\binom{\lambda+i}{i}\int_0^{\infty} x^{r+\alpha-1}e^{-(i+1)\beta x^{\alpha}}dx \tag{15}$$

Let

$$v = \beta\left(i+1\right)x^{\alpha},\ \ x = \left\{\frac{v}{\beta\left(i+1\right)}\right\}^{\frac{1}{\alpha}},\ \Rightarrow\ dx = \frac{dv}{\alpha\beta\left(i+1\right)x^{\alpha-1}} \tag{16}$$

Putting (16) into (15) we can get

$$E\left(x^r\right) = 2\lambda\sum_{i=0}^{\infty}\frac{(-1)^i}{\beta^{\frac{r}{\alpha}}\left(i+1\right)^{1+\frac{r}{\alpha}}}\binom{\lambda+i}{i}\left(1+\frac{r}{\alpha}\right) \tag{17}$$

### 3.2. Quantile Function

The quantile function of the LBIII is derived by inverting (5) as

$$x_q = \left\{-\frac{1}{\beta}\log\left(u^{-\frac{1}{\lambda}}-1\right)\right\}^{\frac{1}{\alpha}}; \qquad u\in\left[0,1\right] \tag{18}$$

where $u$ has a uniform random variable with interval 0 and 1.

## 4. Parameter Estimation

In this section, the parameters of the LBIII distribution will be determined employing the Maximum Likelihood (ML) approach.

Let $x_1, x_2, \ldots x_n$ denote the possible outcomes of a random sample of size n that was drawn from the LBIII model with vector parameter $\boldsymbol{\psi} = \left(\alpha,\beta,\lambda\right)^T$. To determine the ML estimator of the parameter $\boldsymbol{\psi}$, the log-likelihood function of (4) denoted by $\ell$ is given by

$$\ell = n\log\left(\alpha\right)+n\log\left(\beta\right)+n\log\left(\lambda\right)+\left(\alpha-1\right)\sum_{i=1}^{n}\log\left(x_i\right)-\beta\sum_{i=1}^{n}\left(x_i^{\alpha}\right)-\left(\lambda+1\right)\sum_{i=1}^{n}\log\left(1+e^{-\beta x_i}\right) \tag{19}$$

Therefore, the ML estimator $\hat{\boldsymbol{\psi}}$ of $\boldsymbol{\psi}$ can be derived by maximizing (19), this can be done by considering some statistical packages such R-package, and so on.

## 5. Applications

The performance of the novel Transformed Log Burr III distribution is demonstrated in this section by using data from the first birth of 107 SINDI race cows. This distribution can be compared to existing distributions such as the Burr III and New Modified Burr III (NMBIII), and its performance is measured using information criteria such as the Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), and Hannan-Quinn Information Criterion (HQIC). The distribution with the lowest value of those criteria should be considered the best fit for the data set.

*Data Set: Milk Production Data*

This data is presented as follows: The data considered in this study can be found in a study conducted by [15], and it comprises transformed milk production in the first birth of 107 cows from the SINDI race.

The findings of the transformed log BIII distribution were compared to the outcomes of competing distributions in Table 1.

Table 1 displays the estimated, AIC, CAIC, BIC, and HQIC for the proposed distribution as well as the competing distributions. The proposed LBIII model has the lowest values of the AIC, CAIC, BIC, and HQIC. This demonstrates that the LBIII model is the best fit for the milk production data set.

**Table 1.** Results for Milk Production data.

| Model | Estimate | AIC | CAIC | BIC | HQIC |
|---|---|---|---|---|---|
| BIII | $\hat{\lambda}$ = 1.0970 | 163.1852 | 163.3006 | 168.5309 | 165.3523 |
| | $\hat{\beta}$ =1.0946 | | | | |
| NMBIII | $\hat{\theta}$ = 0.2210 | 190.8067 | 191.0398 | 198.8252 | 194.0573 |
| | $\hat{\beta}$ = 0.6682 | | | | |
| | $\hat{\lambda}$ = 1.1411 | | | | |
| LBIII | $\hat{\alpha}$ = 1.1425 | 142.7657 | 142.9988 | 150.7842 | 146.0163 |
| | $\hat{\beta}$ = 1.8952 | | | | |
| | $\hat{\lambda}$ = 1.8509 | | | | |

## 6. Conclusions

In this paper, we propose a novel transformed log BIII distribution as an alternative to the Burr III and New Modified Burr III distributions. The proposed distribution could be symmetrical and left-skewed, with an upside-down bath-tub and decreasing failure rates, and its many features are investigated. The adaptability of the novel distribution was demonstrated using real data sets relating to milk production at the first birth of 107 SINDI race cows, and the findings revealed that this distribution fitted the data set.

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