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Dynamics of Beddington De-Angelis Type Eco-Epidemiological Model with Prey Refuge and Prey Harvesting

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Abstract: Analysing the prey-predator model is the purpose of this paper. In interactions of the Beddington De-Angelis type, the predator consumes its prey. Researchers first examine the existence and local stability of potential unbalanced equilibrium boundaries for the model. In addition,For the suggested model incorporating the prey refuge, we investigate the Hopf-bifurcation inquiry. To emphasise our key analytical conclusions, we show some numerical simulation results at the end.

Keywords: prey refuge; prey harvesting; equilibrium points; stability; Hopf-bifurcation

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1. Introduction

In prey predator models are two type one is an ecological model and another one is an epidemiological model. In ecological model interactions between organisms, including humans, and their physical environment. In epidemiological models are used to study diseces in animals and humans. Also, the above study of ecology and epidemiology is called eco-epidemiology. In 1949, Solomon first used the term 'functional response'. In the late 1950s, C. S. (Buzz) Holling conducted experiments to investigate how predators capture prey. In the resulting series of influential articles, Holling established three main functional response types, which he referred to as Holling types 1, 2, and 3. The Holling type I functional response g(X,Y) = aX, where $a > 0$, is based on the principle of mass action and depends on the prey. Therefore, in the event of a superabundant supply of food, predators will feed at the highest rate possible for each individual predator, and a subsequent rise in food supply will not be able to increase the eating rate further. Because of this, it is given in the form $g(X,Y) = \frac{bX}{w+X}$, which is bounded as well as non-linear (the Michaelis-Menten function or the Holling type II function). Except at low prey density, the Holling type 3 is similar to the type 2, but the Holling type III prey capture rate accelerates. The Holling type III functional response is of the form $g(X,Y) = \frac{cX^2}{w+X^2}$, which is bounded as well as non-linear[6].Up to a certain range, the Holling type II functional response accurately describes feeding rate; however, there may be circumstances in which an increase in predator density indicates a decrease in feeding rate because of mutual interference between individual predators. For this reason, we transform the Holling type II functional response into the Beddington-DeAngelis functional response, $g(X,Y) = \frac{bX}{w_1+Y+w_2X}$ [3]. DeAngelis proposed the Beddington-DeAngelis functional response to solve the apparent problems with the predator-prey model. For describing parasite-host interaction independently, Beddington offered the same kind of functional response. It accurately represents the majority of the qualitative features of the ratio-dependent model while avoiding the "low density problem," which is typically contentious[2–5].The prey refuge and harvesting are incorporated into the eco-epidemiological model using Holling type II behaviour, which has been studied by many authors[1]. In this paper, We analyze the Beddington De-Angelis type eco-epidemiology model's behaviour towards the prey refuge and prey harvesting[7]. This piece is structured as follows: The prey-predator system's past is described as section 1. In section 2, the model formation is presented. Section 3 shows some mathematicals results like positivity, positive invariance and boundedness. The existence of equilibrium points is described in section 4. Local stability analyses in section 5. The Global stability and Hope-Bifurcation Analysis is found in Section 6 and 7. Results are presented numerically in section 8. Finally, this paper concludes with a few observations about the suggested system in secton 9.

2. Model Formation

The non-linear differential equation are:

$$\left.\begin{array}{rcl}
\frac{d\mathcal{S}}{d\mathcal{T}} &=& r_1\mathcal{S}(1-\frac{\mathcal{S}+\mathcal{I}}{\mathcal{L}})-\eta\mathcal{I}\mathcal{S}-\frac{\omega_1\mathcal{S}\mathcal{W}}{\beta_1+\mu\mathcal{S}+\vartheta\mathcal{W}}-\mathcal{E}_1\mathcal{H}_1\mathcal{S},\\
\frac{d\mathcal{I}}{d\mathcal{T}} &=& \eta\mathcal{I}\mathcal{S}-d_1\mathcal{I}-\frac{\gamma_1(1-\theta)\mathcal{I}\mathcal{W}}{\beta_1+(1-\theta)\mathcal{I}}-\mathcal{E}_2\mathcal{H}_2\mathcal{I},\\
\frac{d\mathcal{W}}{d\mathcal{T}} &=& -d_2\mathcal{W}+\frac{c\gamma_1(1-\theta)\mathcal{I}\mathcal{W}}{\beta_1+(1-\theta)\mathcal{I}}+\frac{c\omega_1\mathcal{S}\mathcal{W}}{\beta_1+\mu\mathcal{S}+\vartheta\mathcal{W}}.
\end{array}\right\} \quad (1)$$

and the positive values are $\mathcal{W}>0$,$\mathcal{S}>0$ and $\mathcal{I}>0$.

Table 1. A physiological meanings of parameters are listed in the below chart.

Parameters	Physiological representation	Units
ϑ	Magnitude of interference of predator	m
μ	Effect of handing time for predator	m
$\mathcal{H}_1,\mathcal{H}_2$	The Harvesting effort of predator	No.per unit area(tons)
r_1	Prey growth rate	per day (t^{-1})
L	Environment carrying capacity	No.per unit area(tons)
$\mathcal{E}_2$ and $\mathcal{E}_1$	Catchability coefficient of predator	per day (t^{-1})
β_1	Constant of Half-saturation	m
ω_1	Susceptible Prey rate of Predation	per day (t^{-1})
c	Conversion rate of prey to predator	$0\leq C\leq 1$
d_1 and d_2	Death rate of infected prey and predator	per day (t^{-1})
γ_1	Infected Prey Predation rate	per day (t^{-1})
η	The incidence of contamination for prey	per day (t^{-1})
θ	Refuge of prey	m^{-1}
$\mathcal{W},\mathcal{S}and\mathcal{I}$	Predator, Susceptible and Infected Prey	No.per unit area(tons)

$$\left.\begin{array}{rclr}
\frac{ds}{dt} &=& rs(1-s-i)-si-\frac{\omega sw}{\beta+\mu s+\vartheta w}-h_1s & =f_1(s,i,w),\\
\frac{di}{dt} &=& is-di-\frac{\gamma(1-\theta)iw}{\beta+(1-\theta)i}-h_2i & =f_2(s,i,w),\\
\frac{dw}{dt} &=& -\varphi w+\frac{c\gamma(1-\theta)iw}{\beta+(1-\theta)i}+\frac{c\omega sw}{\beta+\mu s+\vartheta w} & =f_3(s,i,w),
\end{array}\right\} \quad (2)$$

where,reduce parameter is as follows: $w=\frac{\mathcal{W}}{\mathcal{L}},i=\frac{\mathcal{I}}{\mathcal{L}},s=\frac{\mathcal{S}}{\mathcal{L}},t=\eta\mathcal{L}\mathcal{T}$. $r=\frac{r_1}{\eta L},\omega=\frac{\omega_1}{\eta L},h_1=\frac{E_1H_1}{\eta L},d=\frac{d_1}{\eta L},h_2=\frac{E_2H_2}{\eta L},\gamma=\frac{\gamma_1}{\eta L},\beta=\frac{\beta_1}{L},\varphi=\frac{d_2}{\eta L}$, and the initial conditions $w(0)\geq 0,s(0)\geq 0$ and $i(0)\geq 0$. The above-defined functions are in $\mathbb{R}^3_+$.

3. Mathematical Results

3.1. Positive Invariance

Note the function $f_i(s,i,w)$, $i=1,2,3$ are defined for $s>0,i>0,w>0$. $\lim_{(s,i,w)\to(0,0,0)} f_i(s,i,w)=0$, $i=1,2,3$. Using $f_i(0,0,0)$=0, $i=1,2,3$ we can extend the domain and conclude that the functions $f_i(s,i,w)$, $i=1,2,3$ is locally Lipschitzian and continuous on $\mathbb{R}^3_+$ =$\{(s,i,w):s\geq 0,i\geq 0,w\geq 0\}$. Hence, the soluation of equation (2) with non-negative initial condition exists and is unique. It can be show that these solution exists for $t > 0$ and stay non-negative. Hence, the region $\mathbb{R}^3_+$ is invariant for the system (2).

3.2. Positivity of Solutions

Theorem 1. ***The solutions of (2) are positive in the $\mathbb{R}^3_+$.***

Proof. Since $s(0) \geq 0, i(0) \geq 0$ and $w(0) \geq 0$ (2) becomes,

$$\begin{array}{rcl} s(t) & = & s(0)exp\left(\int_0^1 \left[r(1-s-i) - i - \frac{\omega w}{\beta+\mu s+\vartheta w} - h_1\right]dt\right) \geq 0, \\ i(t) & = & i(0)exp\left(\int_0^1 \left[s - d - \frac{\gamma(1-\theta)w}{\beta+(1-\theta)i} - h_2\right]dt\right) \geq 0, \\ w(t) & = & w(0)exp\left(\int_0^1 \left[-\varphi + \frac{c\gamma(1-\theta)i}{\beta+(1-\theta)i} + \frac{c\omega s}{\beta+\mu s+\vartheta w}\right]dt\right) \geq 0. \end{array}$$

Therefore equation (2) are positive in $\mathbb{R}^3_+$. □

3.3. Boundedness of Soluation

Theorem 2. ***The solutions of (2) are bounded in $\mathbb{R}^3_+$.***

Proof. The prey population in the system (2), it is observed that $\frac{ds}{dt} \leq rs(1-s)$. We have, $\lim_{t\to\infty} sup\, s(t) \leq 1$. let $z = s + i + w$

$\frac{dz}{dt} = \frac{ds}{dt} + \frac{di}{dt} + \frac{dw}{dt}$

$= rs(1-s) - \frac{(1-c)\omega s w}{\beta+\mu s+\vartheta w} - h_1 s - (d+h_2)i - \frac{(1-c)\gamma(1-\theta)iw}{\beta+(1-\theta)i} - \varphi w$

$\leq \frac{r}{4} - h_1 s - (d+h_2)i - \varphi w \left(Max, rs(1-s) = \frac{r}{4} and c < 1\right)$

$\leq \frac{r}{4} - \zeta z$, where, $\zeta = min(h_1, d+h_2, \omega)$

We have, $\frac{dz}{dt} + \zeta z \leq \frac{r}{4}$. Using the fundamental concept of differential inequality, we derive $0 < z \leq \frac{r}{4\zeta}(1 - \exp^{-\zeta t}) + z(s_0, i_0, w_0)\exp^{-\zeta t}$. For $t \to \infty$, we have $0 < z < \frac{r}{4\zeta}$.

Thus the soluation (2) are bounded in $\mathbb{R}^3_+$,for all $\epsilon > 0$, $\Omega = \{(s,i,w) \in \mathbb{R}^3_+; s+i+w \leq \frac{r}{4\zeta} + \epsilon\}$ □

4. Equilibrium Points

- The $E_0(0,0,0)$ represents the essence of trivial equilibrium.
- $E_1(\frac{r-h_1}{r}, 0, 0)$is the free of infection and predator free equlibrium its exists for $h_1 < r$.
- $E_2(\bar{s}, \bar{i}, 0,)$ is predator free equlibrium . where, $\bar{i} = \frac{r(1-d-h_2)-h_1}{r+1}$, $\bar{s} = d + h_2$.
- Positive equilibrium is $E^*(s^*, i^*, w^*)$, where, $i^* = \frac{-\varphi\beta(s-d-h_2)+s\gamma(1-\theta)[r-rs-h_1]c}{(1-\theta)[\varphi(s-d-h_2)+\gamma(rs+d+h_2)c]}$, $w^* = \frac{c(s-d-h_2)[(rs+d+h_2)\beta+(1-\theta)(r-rs-h_1)s]}{(1-\theta)[\varphi(s-d-h_2)+\gamma(rs+d+h_2)c]}$ and the s^* is exist unique positive roots of the below polynomical equations, $Us^5 - Vs^4 - Ws^3 - Xs^2 - Ys - Z = 0$, where, $U = c\vartheta r^2 e_3 e_4^2, V = [r\mu e_3 e_4^2 e_5 + \vartheta(Fre_3e_4 - cre_4H)], Z = e_1[Me_4e_3\beta - ce_1\beta(M-\omega)]$, $W = [r\mu e_1 e_3^2 e_4^2 + \beta He_4e_5 + \mu Me_4e_5 + cre_4\omega + \vartheta(FM + re_3e_4G - cre_4M)]$ $X = [re_3e_4^2e_5\beta + \mu e_4e_5H + e_1e_3e_4\beta H + \mu e_1e_3e_4M - F\omega + \vartheta(FM + GH - cre_2^2e_3e_4\beta)]$ $Y = [rae_1e_3^2e_4^2 + \mu e_1e_3e_4H + e_4e_5M\beta - G\omega + \vartheta(GM - ce_2^2aH)], M = e_1[e_2e_3e_4 - \varphi\beta]$, $F = c[r\beta + e_2e_4 + re_1e_4], G = ce_1[\beta - r\beta - e_2e_4], H = e_4[e_2\varphi - re_1e_3 - c\gamma e_2] + (r+1)\varphi\beta, e_1 = d + h_2, e_2 = r - h_1, e_3 = c\gamma - \varphi, e_4 = 1 - \theta, e_5 = \varphi + \gamma cr$.

5. Local Stability Analysis

It is necessary to calculate the Jacobian matrix, which is provided by, in order to evaluate the stability of the system. (2) $J(E) = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$ Where,

$x_{11} = -\frac{(\beta+\vartheta w)\omega w}{(\beta+\mu s+\vartheta w)^2} - i(r+1) - h_1 + r(1-2s), x_{12} = -s(1+r), x_{13} = -\frac{(\beta+\mu w)\omega s}{(\beta+\mu s+\vartheta w)^2}$,

$x_{21} = i, x_{22} = s - d - h_2 - \frac{\beta\gamma w(1-\theta)}{(i(1-\theta)+\beta)^2}, x_{23} = -\frac{i\gamma(1-\theta)}{(i(1-\theta)+\beta)},$
$x_{31} = -\frac{(\vartheta w+\beta)\omega c w}{(\mu s+\vartheta w+\beta)^2}, x_{32} = \frac{\beta c\gamma(1-\theta)w}{(\beta+i(1-\theta))^2}, x_{33} = -\varphi + \frac{ic\gamma(1-\theta)}{\beta+(1-\theta)i} + \frac{(\mu s+\beta)\omega c s}{(\beta+\mu s+\vartheta w)^2}.$

Theorem 3. *If the trivial equilibrium point $E_0(0,0,0)$ is stable, if it is $r < h_1$, then it is unstable.*

Proof. The Jacobian matrix for $E_0(0,0,0)$ is $J(E_0) = \begin{pmatrix} r-h_1 & 0 & 0 \\ 0 & -d-h_2 & 0 \\ 0 & 0 & -\varphi \end{pmatrix},$
The characteristic equation of Jacobian matrix is $J(E_0)$,
$((r-h_1)-\lambda_{01})((-d-h_2)-\lambda_{02})(-\varphi-\lambda_{03}) = 0$, here,$\lambda_{02} < 0, \lambda_{03} < 0$ then the equilibrium point E_0 is stable if it is $r < h_1$, then it is unstable. □

Theorem 4. *If $r(1-d-h_2) < h_1$ and $\frac{c\omega(r-h_1)}{r\beta+\mu(r-h_1)} < \varphi$, the equilibrium point $E_1(\frac{r-h_1}{r},0,0)$ within the infected-free and predator-free regions is stable; otherwise, it is unstable.*

Proof. The Jacobian matrix for E_1 is $J(E_1) = \begin{pmatrix} h_1 - r & \frac{-r+h_1}{r}(r+1) & \frac{-\omega(r-h_1)}{r\beta+\mu(r-h_1)} \\ 0 & 1-d-h_2-\frac{h_1}{r} & 0 \\ 0 & 0 & \frac{c\omega(r-h_1)}{r\beta+\mu(r-h_1)} - \varphi \end{pmatrix}$
The characteristic equation of Jacobian matrix is $J(E_1)$,
$(h_1 - r - \lambda_{11})(1-d-h_2-\frac{h_1}{r}-\lambda_{12})(\frac{c\omega(r-h_1)}{r\beta+\mu(r-h_1)} - \varphi - \lambda_{13}) = 0$
here, if $r(1-d-h_2) < h_1$ and $\frac{c\omega(r-h_1)}{r\beta+\mu(r-h_1)} < \varphi$, the equilibrium point $E_1(\frac{r-h_1}{r},0,0)$ within the infected-free and predator-free regions is stable; otherwise, it is unstable. □

Theorem 5. *The Predator free equlibrium point $E_2(d+h_2, \frac{r(1-d-h_2)-h_1}{r+1}, 0)$ is locally asymptotically stable if $\varphi > c(\omega+\gamma)$.*

Proof. The Jacobian matrix at E_2 is $J(E_2) = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$ where,
$b_{11} = -r(d+h_2), b_{12} = -(d+h_2)(r+1), b_{13} = \frac{-\omega(d+h_2)}{\beta+\mu(d+h_2)}, b_{21} = \frac{r(1-d-h_2)-h_1}{r+1},$
$b_{22} = 0, b_{23} = -\frac{i\gamma(1-\theta)}{(\beta+(1-\theta)i)}, b_{31} = 0, b_{32} = 0, b_{33} = -\varphi + \frac{(\beta+\mu s)\omega c s}{\beta+\mu s+\vartheta w} + \frac{ci\gamma(1-\theta)}{\beta+(1-\theta)i}.$
The characteristic equation of Jacobian matrix is $J(E_2)$, $\lambda^3 + R\lambda^2 + Q\lambda + P = 0$, here, $R = -u_{11} - u_{33}, Q = -u_{21}u_{12} + u_{33}u_{11}, P = u_{12}u_{21}u_{33}$. If P,R and RQ-P are positive, According to the Routh-hurwitz criterion, the negative real parts are the root of the above characteristic equation if and only if P,R and RQ-P are positive. $RQ - P = u_{11}u_{33}(-u_{11} - u_{33}) + u_{11}u_{12}u_{21}$. The sufficient conditions for u_{33} to be negative are $\varphi > c(\omega+\gamma)$. hence, $E_2(d+h_2, \frac{r(1-d-h_2)-h_1}{r+1}, 0)$ is locally asymptotically stable. □

Theorem 6. *Locally stable and displaying asymptotic stability, the positive equilibrium point $E*$. If $G > 0, C > 0$ and $GD - C > 0$ .Where $G = -v_{11} - v_{22}, D = -v_{21}v_{12} + v_{22}v_{11} - v_{13}v_{31} - v_{23}v_{32}, C = v_{13}(v_{22}v_{31} - v_{21}v_{32}) + v_{23}(v_{11}v_{32} - v_{12}v_{31})$.*

Proof. At $E*$, the Jacobian matrix is $J(E^*) = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}$ where,
$v_{11} = -\frac{(\beta+\vartheta w)\omega w}{(\beta+\mu s+\vartheta w)^2} - i(r+1) - h_1 + r(1-2s), v_{12} = -s(1+r), v_{13} = -\frac{(\beta+\mu w)\omega s}{(\beta+\mu s+\vartheta w)^2},$
$v_{21} = i, v_{22} = s - d - h_2 - \frac{\beta\gamma w(1-\theta)}{(i(1-\theta)+\beta)^2}, v_{23} = -\frac{i\gamma(1-\theta)}{(i(1-\theta)+\beta)},$
$v_{31} = -\frac{(\vartheta w+\beta)\omega c w}{(\beta+\mu s+\vartheta w)^2}, v_{32} = \frac{\beta c\gamma(1-\theta)w}{(\beta+i(1-\theta))^2}, v_{33} = 0.$

The characteristic equation of Jacobian matrix is$J(E^*)$,

$$\lambda^3 + G\lambda^2 + D\lambda + C = 0 \tag{3}$$

here, G=$-v_{11} - v_{22}$, D=$-v_{21}v_{12} + v_{22}v_{11} - v_{13}v_{31} - v_{23}v_{32}$, C=$v_{13}(v_{22}v_{31} - v_{21}v_{32})$ $+v_{23}(v_{11}v_{32} - v_{12}v_{31})$. If $G > 0, C > 0, GD - C > 0$. According to the Routh-hurwitz criterion, the negative real parts are the root of the above characteristic equation if and only if G,C and GD-C are positive. hence, $E*$ is locally asymptotically stable. □

6. Global Stability Analysis

Theorem 7. *The endemic equilibrium point E^* is globally asymptotically stable.*

Proof. Consider a Lyapunov function

$$V(s,i,w) = \left[s - s^* - s^* ln\frac{s}{s^*}\right] + d_1\left[i - i^* - i^* ln\frac{i}{i^*}\right] + d_2\left[w - w^* - w^* ln\frac{w}{w^*}\right]$$

$$\begin{aligned}
&\tfrac{dV}{dt} = \left[\frac{s-s^*}{w}\right]s(t) + d_1\left[\frac{i-i^*}{i}\right]i(t) + d_2\left[\frac{w-w^*}{w}\right]w(t)\\
&\leq \left(\frac{s-s^*}{s}\right)\left[rs(1-s-i) - si - \frac{\omega sw}{\beta+\mu s+\vartheta w} - h_1 s\right] + d_1\left(\frac{i-i^*}{i}\right)\left[is - di - \frac{\gamma(1-\theta)iw}{\beta+(1-\theta)i} - h_2 i\right]\\
&+d_2\left(\frac{w-w^*}{w}\right)\left[-\varphi w + \frac{c\gamma(1-\theta)iw}{\beta+(1-\theta)i} + \frac{c\omega sw}{\beta+\mu s+\vartheta w}\right]\\
&\leq -(s-s^*)[r(s+i) - r(s^*+i^*) + (i-i^*)] - \omega\left[\frac{w}{\beta+\mu s+\vartheta w} - \frac{w^*}{\beta+\mu s^*+\vartheta w^*}\right]\\
&-d_1(i-i^*)\left[-(s-s^*) + \gamma(1-\theta)\left(\frac{w}{\beta+(1-\theta)i} - \frac{w^*}{\beta+(1-\theta)i^*}\right)\right]\\
&-d_2(w-w^*)\left[-c\omega\left(\frac{s}{\beta+\mu s+\vartheta w} - \frac{s^*}{\beta+\mu s^*+\vartheta w^*}\right) - c\gamma(1-\theta)\left(\frac{w}{\beta+(1-\theta)i} - \frac{w^*}{\beta+(1-\theta)i^*}\right)\right]
\end{aligned}$$

Obviously, $V(s,i,w) \leq 0$. we conclude that E^* is globally asymptotically stable. □

7. Hopf-Bifurcation Analysis

Theorem 8. *If the critical value for the bifurcation parameter θ is exceeded, the model (2) experience the hope-bifurcation. the existence of the following hope-bifurcation criteria at $\theta = \theta^*$, 1.$N_1(\theta^*)N(\theta^*) - N_3(\theta^*) = 0$. 2.$\frac{d}{d\theta}(Re(\lambda(\theta)))|_{\theta=\theta^*} \neq 0$ Where, λ is the characteristic of the equation's naught in reference to its underlying a state of equilibrium position.*

Proof. For $\theta = \theta^*$, (3) is in the form of an attribute equation.

$$(\lambda^2(\theta^*) + N_2(\theta^*))(\lambda(\theta^*) + N_1(\theta^*)) = 0. \tag{4}$$

Which implies that $\pm i\sqrt{N_2(\theta^*)}$ and $-N_1(\theta^*)$ be the zero of the above equation. The following transversality requirement must be satisfied in order for us to achieve the Hopf-bifurcation at $\theta = \theta^*$. $\frac{d}{d\theta^*}(Re(\lambda(\theta^*)))| \neq 0$. The general solutions of the previously mentioned equation for all θ. (4) λ_1=$r(\theta)$+ $is(\theta)$, λ_2=$r(\theta)$- $is(\theta)$, λ_3=$-N_1(\theta)$. We have, $\frac{d}{d\theta^*}(Re(\lambda(\theta^*)))| \neq 0$. Let $\lambda_1 = r(\theta) + is(\theta)$ in the (4), we get $\mathcal{A}(\theta) + i\mathcal{B}(\theta) = 0$. Where, $\mathcal{A}(\theta)$=$r^3(\theta)$+$r^2(\theta)N_1(\theta)$-$3r(\theta)s^2(\theta) - s^2(\theta)N_1(\theta)$+$N_2(\theta)r(\theta) + N_1(\theta)N_2(\theta)$, $\mathcal{B}(\theta)$=$V_2(\theta)s(\theta)$+ $2r(\theta)s(\theta)N_1(\theta)$+$3r^2(\theta)N(\theta)$+$s^3(\theta)$.

$$\frac{d\mathcal{A}}{d\theta} = \chi_1(\theta)r'(\theta) - \chi_2(\theta)s'(\theta) + \chi_3(\theta) = 0, \tag{5}$$

$$\frac{d\mathcal{B}}{d\theta} = \chi_2(\theta)r'(\theta) + \chi_1(\theta)s'(\theta) + \chi_4(\theta) = 0. \tag{6}$$

where, χ_1=$3r^2(\theta)$+$2r(\theta)N_1(\theta)$-$3s^2(\theta)$+$N_2(\theta)$,χ_2=$6r(\theta)s(\theta)$+$2s(\theta)N_1(\theta)$, χ_3=$r^2(\theta)N_1^{'}(\theta)$+$s^2(\theta)N_1^{'}(\theta)$+$N_2^{'}(\theta)r(\theta)$, χ_4=$N_2^{'}(\theta)s(\theta)$+$2r(\theta)s(\theta)N_1^{'}(\theta)$.
On multiplying (5) by $\chi_1(\theta)$ and (6) by $\chi_2(\theta)$ respectively

$$r(\theta)^{'} = -\frac{\chi_1(\theta)\chi_3(\theta) + \chi_2(\theta)\chi_4(\theta)}{{\chi_1}^2(\theta) + {\chi_2}^2(\theta)}. \tag{7}$$

Substituting $r(\theta) = 0$ and $s(\theta) = \sqrt{N_2(\theta)}$ at $\theta = \theta^*$ on $\chi_1(\theta), \chi_2(\theta), \chi_3(\theta)$, and $\chi_4(\theta)$, we obtain $\chi_1(\theta^*)$=$-2N_2(h_2^*)$, $\chi_2(\theta^*)$=$2N_1(\theta^*)\sqrt{N_2(\theta^*)}$, $\chi_3(\theta^*)$=$N_3^{'}(\theta^*)$-$N_2(l^*)N_1^{'}(\theta^*)$, $\chi_4(\theta^*)$=$N_2^{'}(\theta^*)\sqrt{N_2\theta^*}$. The equation (7), implies $r^{'}(\theta^*) = \frac{N_3^{'}(\theta^*)-(N_1(\theta^* N_2(\theta^*)))}{2(N_2(\theta^*)+N_1^2(\theta^*))}$, if $N_3^{'}(\theta^*) - (N_1(l]\theta^*)N_2(\theta^*))^{'} \neq 0$ it suggests that $\frac{d}{d\theta^*}(Re(\lambda(\theta^*)))| \neq 0$, and $\lambda_3(\theta^*) = -N_1(\theta^*) \neq 0$. Therefore the conditios $N_3^{'}(\theta^*) - (N_1(\theta^*)N_2(\theta^*))^{'} \neq 0$ is ensured that the transversality requirement holds, as a result, the model (2) has entered the Hopf bifurcation at $\theta = \theta^*$. ☐

8. Numerical Simulations

In this section, we perform some numerical simulations on the system (2)in order to verify our theoretical findings.In this study, susceptible prey predator rate (ω) and prey refuge (θ) well be taken as important control parameters. Through the use of the MATLAB software, each of us performed various mathematical simulations with these particular parameter values are r=0.5, β= 0.2, d= 0.1, c= 0.5, γ= 0.1, h_2= 0.2, ϑ= 0.3, μ= 0.2, φ= 0.12,h_1= 0.01, γ= 0.1, ω= variable, θ= variable [8].

8.1. Effect of Varying the Susceptible Prey Predator Rate ω

We should adjust the database variable 2 as $\theta = 0.2$. For the given limitation value, E^* is stable at positive equilibrium point $\omega \geq 0.3$.

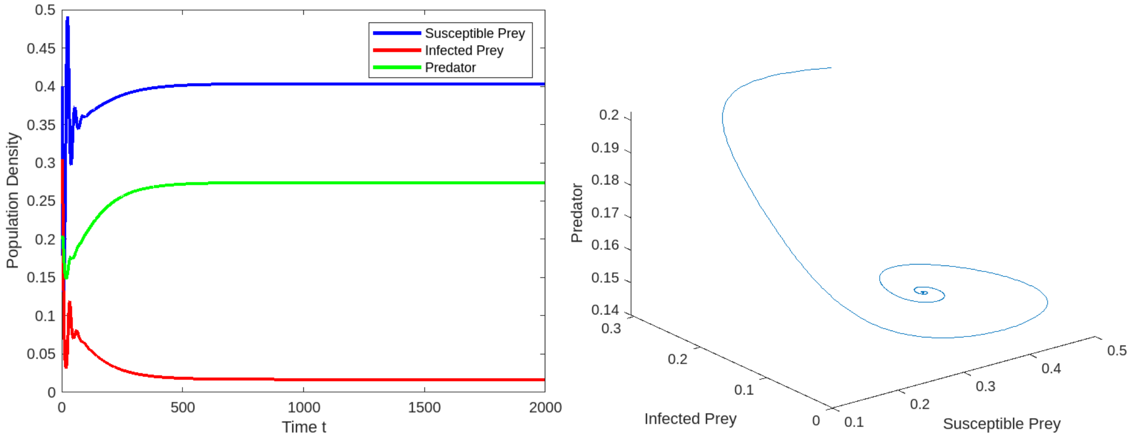

Figure 1. The system's time series solution additionally parametric plot are displayed in the previously mentioned figure. (2) with limitation values in Table 2 except $\theta = 0.2$ and $\omega = 0.35$.

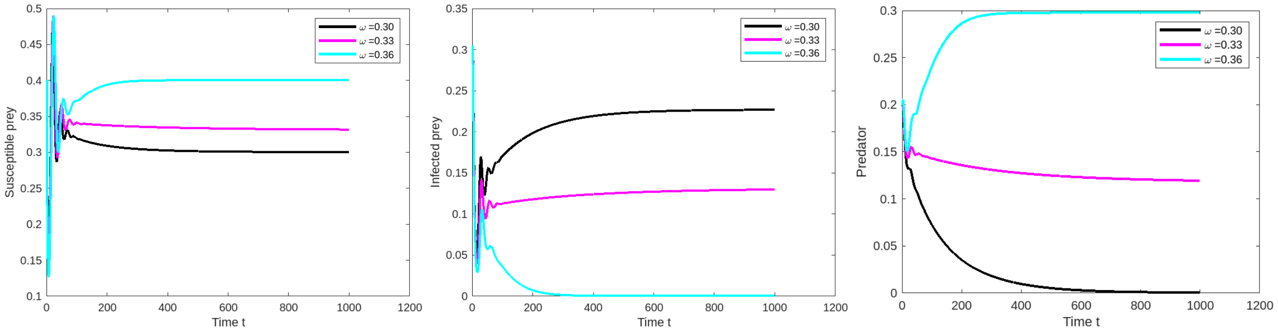

Figure 2. The compactness of predator population, infected and susceptible prey for the limitation values in table 2 except $\theta = 0.2$ and $\omega = 0.30, 0.33$ and 0.36.

8.2. Effect of Varying the Prey Refuge θ

We should adjust the database variable 2 as $\omega = 0.3$. For the given limitation value, E^* is stable at positive equilibrium point for $\theta \geq 0.2$.

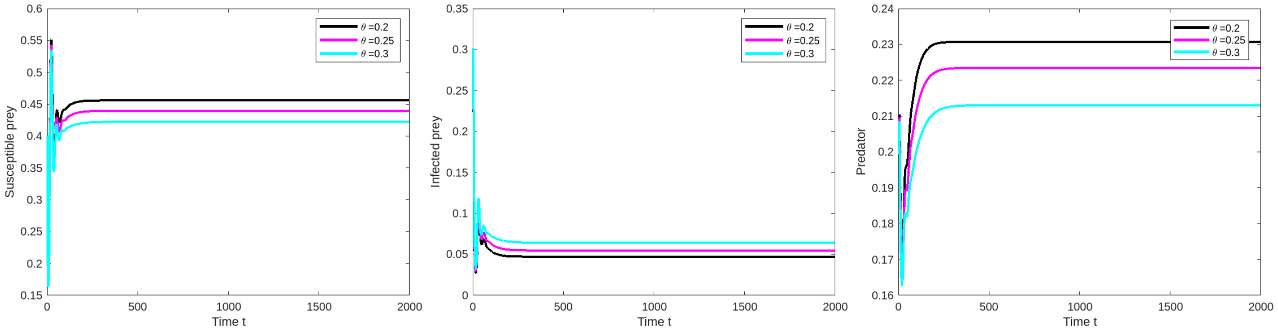

Figure 3. The compactness of predator poulation, infected and susceptible prey for the limitation values in Table 2 except for $\omega = 0.3$ and $\theta = 0.2, 0.25$ and 0.3.

9. Conclusions

In this study, we investigated the three- species food web eco-epidemiolodical model with prey refuge in infected prey population and harvesting sffect in both prey populations. Some mathsmatical results like positive invariance, positivity and boundedness analysed in system (2). The local stability is assigned to each biologically feasible equilibrium point of the system.Golbal stability analysed by sutiable lyapunov function. Hofe-bifurcation analysed by bifurcation parameter (θ).Also, Prey refuge (θ) and susceptible prey predator rate (ω) is used as acontrol parameter. According to the analytical and numerical findings, the prey refuge and susceptible prey predator rate has a major impact on the population. If we increase the susceptible prey predator rate and prey refuge in predator populations, the system loses its stability. This study shows the complex behavior of the proposed model.

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