engineering
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Proceeding Paper

Dynamics and Bifurcation Analysis of an Eco-Epidemiological Model in Crowley-Martin Functional Response with the Impact of Fear †

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† Presented at the 4th International Electronic Conference on Applied Sciences, 27 October–10 November 2023; Available online: https://asec2023.sciforum.net/.

Abstract: This article consists of a three-species food web model that has been developed by considering the interaction between susceptible prey, infected prey, and predator species. It is assumed that susceptible prey species grow logistically in the absence of predators. It is assumed that predators consume susceptible and infected prey and infected prey consumes susceptible prey. We consider the effect of fear on susceptible prey due to predator species. Furthermore, the predator consumes its prey in the form of Holling-type and Crowley-Martin-type interactions. Also, infected prey consumes susceptible prey in the form of Holling-type interaction. The conditions of all biologically feasible equilibrium points have been examined. The local stability of the systems around these equilibrium points is investigated Furthermore, the occurrence of Hopf-bifurcation concerning fear ϱ of the system has been investigated. Finally, we demonstrate some numerical simulation results to illustrate our main analytical findings.

Keywords: infected prey; fear effect; Crowley-Martin; equilibrium point; stability; bifurcation

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Citation: M, S.P.; T, N.G.; A, Y. Dynamics and Bifurcation Analysis of an Eco-Epidemiological Model in Crowley-Martin Functional Response with the Impact of Fear. *Eng. Proc.* **2023**, *52*, 0. https://doi.org/

Academic Editor: Firstname Lastname

Published:

1. Introduction

Eco-epidemiological systems are used to investigate the dynamic connection between predator and prey in one population or a population of susceptible and infected animals. Mathematical models have become significant instruments in examining the flow and manipulation of prevention. Since Kermack-Mckendrick's pioneering work on SIRS [1], epidemiological models have drawn a lot of interest from researchers. Ecology and epidemiology are two distinct essential and significant areas of research. Lotka [2] and Volterra [3] models, The first advance in current mathematical ecology can be examined using the system of dynamical equations. It is referred to as the study of infection spread between interacting organisms. A biological representation in terms of mathematical modelling of communications among the populations density of predators and population density of prey, called "functional response". Modelling in biological systems There are numerous of functional responses namely the Holling type [4,5], type of Beddington-DeAngelis responses, type of Crowley-Martin responses; Arditi and Ginzburg's [6] much more information on predator-prey systems with Crowley-Martin functional responses has become available in recent decades. In the recent era, some renowned authors [7]. They used some functional responses such as type of Crowley-Martin functional response to make the model system, more realistic and controllable in the eco-system. To the best of our knowledge, no one has examined a three-species food web eco-epidemiological model with Holling type I, II, and Crowley-Martin functional responses, along with the impact of fear and disease on prey populations. Motivated by this fact, we explore a three-species food web eco-epidemiological model with Holling type I, Holling type II, and Crowley-Martin functional responses to fear in susceptible prey populations due to predator species. The occurrence of Hopf bifurcation analysis for the proposed model in relation to the existence of the fear effect. The rest of the paper is structured as follows: In Section 2, we present the mathematical analysis that has been investigated. Section 3 deals with the point of equilibrium in boundary and their stability. In Section 4 we determine the existence of the interior point of equilibria $E^*(s^*, i^*, p^*)$ and investigate its local stability. The occurrence of Hopf-bifurcation is shown in Section 5. Numerical simulations are examined for the proposed model in Section 6. Section 7, which concludes the paper.

2. Model Formation

The framework demonstrates the relationship between the population density of prey with infection. Which leads to the following structure of non-linear differential equations. The suggested framework was applied to examine the non-linear population density of susceptible, infected prey and predator biological model,

$$\left.\begin{array}{rcc} \frac{d\mathcal{S}}{dT} &=& r_1\mathcal{S}(1-\frac{\mathcal{S}+\mathcal{I}}{\mathcal{K}}) - \lambda\mathcal{I}\mathcal{S} - \frac{\alpha_1\mathcal{S}\mathcal{P}}{(1+\zeta\mathcal{S})(1+\eta\mathcal{P})}, \\ \frac{d\mathcal{I}}{dT} &=& \lambda\mathcal{I}\mathcal{S} - d_1\mathcal{I} - \frac{b_1\mathcal{I}\mathcal{P}}{a_1+\mathcal{I}}, \\ \frac{d\mathcal{P}}{dT} &=& -d_2\mathcal{P} + \frac{cb_1\mathcal{I}\mathcal{P}}{a_1+\mathcal{I}} + \frac{c\alpha_1\mathcal{S}\mathcal{P}}{(1+\zeta\mathcal{S})(1+\eta\mathcal{P})}. \end{array}\right\} \tag{1}$$

In the above biological systems the susceptible prey population fear about the popukation of predator. The reproduction rate of susceptible prey population will be decreases due to fear on predator. Here the conditions are $\mathcal{S}(0) \geq 0, \mathcal{I}(0) \geq 0$ and $\mathcal{P}(0) \geq 0$. The condition for the fear effect is

$$\mathcal{F}_1(\varrho, p) = \frac{1}{1+\varrho p} \tag{2}$$

This describes the level of fear in susceptible prey as a consequence of the predator. Here, ϱ represents the quantity of fear. Given the epidemiological meaning of ϱ, the following condition is strongly acceptable:

$$\varrho(0,p) = \mathcal{F}_1(\varrho,0) = 1, lim\varrho \to \infty\mathcal{F}_1(\varrho,p) = 0 = limp \to \infty\mathcal{F}_1(\varrho,p)$$
$$\frac{\partial\mathcal{F}_1(\varrho,p)}{\partial\varrho} < 0, \frac{\partial\mathcal{F}_1(\varrho,p)}{\partial p} < 0.$$

In this work we incorporate prey and the fear effect ϱ. Then the system as follows.

$$\left.\begin{array}{rcc} \frac{d\mathcal{S}}{dT} &=& \frac{r_1\mathcal{S}}{1+\mathcal{F}\mathcal{P}}(1-\frac{\mathcal{S}+\mathcal{I}}{\mathcal{K}}) - \lambda\mathcal{I}\mathcal{S} - \frac{\alpha_1\mathcal{S}\mathcal{P}}{(1+\zeta\mathcal{S})(1+\eta\mathcal{P})}, \\ \frac{d\mathcal{I}}{dT} &=& \lambda\mathcal{I}\mathcal{S} - d_1\mathcal{I} - \frac{b_1\mathcal{I}\mathcal{P}}{a_1+\mathcal{I}}, \\ \frac{d\mathcal{P}}{dT} &=& -d_2\mathcal{P} + \frac{cb_1\mathcal{I}\mathcal{P}}{a_1+\mathcal{I}} + \frac{c\alpha_1\mathcal{S}\mathcal{P}}{(1+\zeta\mathcal{S})(1+\eta\mathcal{P})}. \end{array}\right\} \tag{3}$$

The Table 1 displays specific biological meanings of the parameters.

In system (3) has many parameters with different units its inconvenient to solve the systems (3), so in our convenient we reduce the system in to non-dimensional equations using the following transformations Here, $s = \frac{\mathcal{S}}{\mathcal{K}}, i = \frac{\mathcal{I}}{\mathcal{K}}, p = \frac{\mathcal{P}}{\mathcal{K}}$, with non-dimensional time $t = \lambda\mathcal{K}T$ Now the (3) becomes,

$$\left.\begin{array}{rcc} \frac{ds}{dt} &=& \frac{rs}{1+\varrho p}(1-s-i) - is - \frac{s\alpha p}{(1+\zeta s)(1+\eta p)} \\ \frac{di}{dt} &=& is - di - \frac{\theta ip}{a+i} \\ \frac{dp}{dt} &=& -\delta p + \frac{c\theta ip}{a+i} + \frac{c\alpha sp}{(1+\zeta s)(1+\eta p)}. \end{array}\right\} \tag{4}$$

here the conditions are, $r = \frac{r_1}{\lambda\mathcal{K}}, \alpha = \frac{\alpha_1}{\lambda\mathcal{K}}, d = \frac{d_1}{\lambda\mathcal{K}}, \theta = \frac{b_1}{\lambda\mathcal{K}}, \delta = \frac{d_2}{\lambda\mathcal{K}}, \varrho = \frac{\mathcal{F}}{\mathcal{K}}$. According to the preliminary criteria $\{s(0), i(0), p(0)\} \geq 0$. The described over are in $\mathbb{R}^3_+$.

Table 1. Specific biological meanings of the parameters (3).

Parameters	Units	Physiological Representation
$\mathcal{S}$	Components per unit area (tons)	Population density of susceptible Prey
$\mathcal{I}$	Components per unit area (tons)	Population density of prey with infection
$\mathcal{P}$	Components per unit area (tons)	Population density of Predator
r_1	Per day (T^{-1})	Prey population densities growth rate
$\mathcal{K}$	Components per unit area (tons)	The carrying
λ	Per day (T^{-1})	Infection rate
a	Per day (T^{-1})	Constant of Half-saturation
α_1	Per day (T^{-1})	Susceptible prey to predator consumption
b_1	Per day (T^{-1})	Capture rate by predator
c	Per day (T^{-1})	Conversion rate of prey to predator
d_1, d_2	Per day (T^{-1})	Diseased prey and predator death rate
$\mathcal{F}$	Components per unit area (tons)	Impact of fear
ζ, η	Per day(T^{-1})	Constant of feeding rate

3. The Existence Point of Equilibrium

The system (4) has three points of equilibrium and one endemic point of equilibrium.

- The $E_0(0,0,0)$ is the point of equilibrium, which is trivial,
- $E_1(\frac{r-1}{r},0,0)$ be the free of infection and free of predator point of equilibrium its exists for $r > 1$.
- The absence of predator point of equilibrium is $E_2(\hat{s},\hat{\imath},0)$,
  where, $\hat{s} = d+2, \hat{\imath} = \frac{r(1-d-2)-1}{r+1}$, its exists for $r(1-2-d) > 1$
- endemic equilibrium is $E^*(s^*,i^*,p^*)$, where,$i^* = \frac{a(a\delta+(\delta-c\alpha)s^*)}{(c\alpha s^*+(c\theta-\delta)(1+\zeta s^*)(1+\eta p^*))}$,
  $p^* = \frac{ac(s^*-d)(1+\zeta s^*)}{(c\alpha s^*+(c\theta-\delta)(1+\zeta s^*))}$, and the $s^*$is the quadratic equation's one and only positive root, $\mathcal{A}S^2 + \mathcal{B}S + \mathcal{C} = 0$, where,

$$\mathcal{A} = r(\alpha c + \theta c - \delta), \mathcal{B} = (\theta c - \delta)(ar - r) + \alpha c((1+\varrho p) - r) + a(\delta(1+\varrho p) + (\delta - c\alpha)r),$$
$$\mathcal{C} = -a(r(1+\varrho p))(c\theta - \delta) + (c\alpha(1+\varrho p)(d) - a\delta((1+\varrho p) + r))).$$

If endemic equilibrium exist for $\delta > \alpha c, r > 1, s^* - d > \frac{(1+r)a\delta}{a\alpha}$, and $a\delta + s^*(\delta - \alpha c)$.

4. Local Stability Analysis

I. We begin by determining the system's (4) Jacobian matrix. $J(E) = \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{pmatrix}$.

Where,

$$n_{11} = \frac{r}{1+\varrho p}(1-2s) - i\left(\frac{r}{1+\varrho p} + 1\right) - \frac{\alpha p}{(1+\zeta s)^2(1+\eta p)}, n_{12} = -s(\frac{r}{1+\varrho p} + 1),$$
$$n_{13} = \frac{prs}{(1+\varrho p)^2}(1-s-i) - \frac{\alpha s}{(1+\zeta s)(1+\eta p)^2}, n_{21} = i, n_{22} = s - d - \frac{a\theta p}{(a+i)^2},$$
$$n_{23} = -\frac{\theta i}{(a+i)}, n_{31} = \frac{c\alpha p}{(1+\zeta s)^2(1+\eta p)}, n_{32} = \frac{ac\theta p}{(a+i)^2}, n_{33} = -\delta + \frac{c\theta i}{a+i} + \frac{\alpha cs}{(1+\zeta s)(1+\eta p)^2}.$$

Theorem 1.
- $E_0(0,0,0)$ *is the trivial equilibrium point is locally stable if* $r < 1$, *otherwise unstable.*
- $E_1(\frac{r-1}{r},0,0)$ *is an infection-free and predator-free equilibrium point is locally stable if* $c\alpha < \delta$ *and* $1 > r(1-d-2)$, *otherwise unstable.*

Proof. The characteristic equation of the point of equilibrium E_0 is $(\lambda_{01} - (r-1))(\lambda_{02} - (-d-2))(\lambda_{03} + \delta) = 0$, $\lambda_{01} = r-1, \lambda_{02} = -d-2, \lambda_{03} = -\delta$. Here, $\lambda_{02} < 0, \lambda_{03} < 0$ $E_0(0,0,0)$ is the trivial equilibrium point. It is locally stable if $r < 1$ is otherwise unstable.

The characteristic equation of the point of equilibrium E_1 is, $(\lambda_{11} - ((1-r)))(\lambda_{12} - (1-d-2-\frac{1}{r}))(\lambda_{13} - (\frac{-\alpha(r-1)}{ra+(r-1)} - \delta)) = 0, \lambda_{11} = 1-r, \lambda_{12} = 1-d-2-\frac{1}{r}, \lambda_{13} = \frac{-c\alpha(r-1)}{ra+(r-1)} - \delta$. Here, $E_1(\frac{r-1}{r}, 0, 0)$ is infection-free and the predator-free equilibrium point is locally stable if $c\alpha < \delta$ and $1 > r(1-d-2)$, otherwise unstable. □

Theorem 2. *The equilibrium* $E_2(\hat{s}, \hat{\imath}, 0)$ *which absence of predator is asymptotically stable if* $\delta > c(\theta + \alpha)$.

Proof. The matrix in the form of Jacobian at E_2 is $J(E_2) = \begin{pmatrix} o_{11} & o_{12} & o_{13} \\ o_{21} & o_{22} & o_{23} \\ o_{31} & o_{32} & o_{33} \end{pmatrix}$,

where,

$$o_{11} = r(1-2\hat{s}) + i(r+1), o_{12} = (-1-r)\hat{s}, o_{13} = -\frac{\alpha\hat{s}}{(1+\zeta\hat{s})}, o_{21} = \hat{\imath}, o_{22} = s-d-2,$$

$$o_{23} = -\frac{\theta\hat{\imath}}{a+\hat{\imath}}, o_{31} = 0, o_{32} = 0, o_{33} = \frac{c\alpha\hat{s}}{1+\zeta\hat{s}} - \delta + \frac{c\theta\hat{\imath}}{a+\hat{\imath}}.$$

The E_2 characteristic equation is, $\lambda^3 + \mathcal{T}\lambda^2 + \mathcal{U}\lambda + \mathcal{V} = 0$. Here,

$$\mathcal{T} = -o_{11} - o_{33},\ \mathcal{U} = -o_{21}o_{12} + o_{33}o_{11},\ \mathcal{V} = o_{12}o_{21}o_{33}.$$

According to the Routh-Hurwitz criterion, if and only if $\mathcal{T}, \mathcal{V}$ and $\mathcal{T}\mathcal{U} - \mathcal{V}$ are non-negative, then the real parts are non-positive roots of the above characteristic equation. Now $\mathcal{T}\mathcal{U} - \mathcal{V} = -o_{11}(-o_{12}o_{21} + o_{33}(o_{33} + o_{11}))$. Now, the necessary criteria for o_{33} to be non-positive is $\delta > c(\alpha + \theta)$. If the above condition in the Theorem is satisfied, the E_2 is locally asymptotically stable. □

Theorem 3. *The endemic or positive point of equilibrium* E^* *is asymptotically stable.*

Proof. The matrix in the form of Jacobian at E^* is $J(E^*) = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$, where,

$$r_{11} = -\frac{s^*(1-r+ar+(1+r)i^*+2rs^*)}{(1+\zeta s^*)^2(1+\eta p^*)}, r_{12} = -s^*(\frac{r}{1+\varrho p^*} + 1),$$

$$r_{13} = \frac{p^* r s^*}{(1+\varrho p^*)^2}(1-s^*-i^*) - \frac{\alpha s^*}{(1+\zeta s^*)(1+\eta p^*)}, r_{21} = i^*,$$

$$r_{22} = \frac{a\theta p^* i^*}{(a+i^*)^2}, r_{23} = \frac{\theta i^*}{(a+i^*)}, r_{31} = \frac{c\alpha p^*}{((1+\zeta s^*)^2(1+\eta p^*))}, r_{32} = \frac{ac\theta p^*}{(a+i^*)^2}, r_{33} = 0.$$

The E^* characteristic equation is

$$\lambda^3 + \mathcal{F}\lambda^2 + \mathcal{G}\lambda + \mathcal{H} = 0, \tag{5}$$

here, $\mathcal{F} = -r_{11} - r_{33}, \mathcal{G} = -r_{21}r_{12} + r_{22}r_{11} - r_{13}r_{31} + r_{23}r_{32}$, $\mathcal{H} = r_{13}(-r_{22}r_{31} + r_{21}r_{32}) + r_{23}(r_{12}r_{31} - r_{11}r_{32})$. If $\mathcal{F} > 0, \mathcal{H} > 0, \mathcal{F}\mathcal{G} - \mathcal{H} > 0$. According to the Routh-Hurwitz criterion, if and only if $\mathcal{F}, \mathcal{H}, \mathcal{F}\mathcal{G} - \mathcal{H}$ are non-negative, then the real parts are non-positive roots of the above characteristic equation. The E^* is locally asymptotically stable. □

5. Hopf-Bifurcation Analysis

The periodic solutions arise or depart due to changes in system parameters, which is called Hopf-bifurication. The eigenvalues of the Jacobian matrix have a negative real part with a complex conjugate, which means bifurication can occur.

Theorem 4. *If the bifurcation parameter* ϱ *exceeds a critical point, the model (4) approaches Hope-bifurcation. At* $\varrho = \varrho^*$*, the following hope-bifurcation conditions arise:*
1. $\mathcal{A}_1(\varrho^*)A(\varrho^*) - \mathcal{A}_3(\varrho^*) = 0.$
2. $\frac{d}{df}(Re(\lambda(\varrho)))|_{\varrho=\varrho^*} \neq 0$ *Here* λ *is the root of the parametric solution correlated with the equilibrium interior point.*

Proof. For $\varrho = \varrho^*$, the characteristic (5) is in the form

$$(\lambda^2(\varrho^*) + \mathcal{A}_2(\varrho^*))(\lambda(^*) + \mathcal{A}_1(\varrho^*)) = 0. \tag{6}$$

This indicates that the roots of the preceding equation are $\pm i\sqrt{\mathcal{A}_2(\varrho^*)}$ and $-\mathcal{A}_1(\varrho^*)$. To achieve the Hopf-bifurcation at $\varrho = \varrho^*$ the following transversality criterion must be fulfilled.

$$\tfrac{d}{d\varrho^*}(Re(\lambda(\varrho^*)))| \neq 0.$$

For ϱ, the above Equation (6) has general roots

$$\lambda_1 = r(\varrho) + is(\varrho), \lambda_2 = r(\varrho) - is(\varrho), \lambda_3 = -\mathcal{A}_1(\varrho).$$

Weather check the criteria $\frac{d}{d\varrho^*}(Re(\lambda(\varrho^*)))| \neq 0$. Let $\lambda_1 = r(\varrho) + is(\varrho)$ in the (6), we get $\mathcal{C}(\varrho) + i\mathcal{D}(\varrho) = 0$. Where,

$$\begin{aligned}
\mathcal{C}(\varrho) &= r^3(\varrho) + r^2(\varrho)\mathcal{A}_1(\varrho) - 3r(\varrho)s^2(\varrho) - s^2(\varrho)\mathcal{A}_1(\varrho) + \mathcal{A}_2(\varrho)r(\varrho) + \mathcal{A}_1(\varrho)\mathcal{A}_2(\varrho),\\
\mathcal{D}(\varrho) &= \mathcal{A}_2(\varrho)s(\varrho) + 2r(\varrho)s(\varrho)\mathcal{A}_1(\varrho) + 3r^2(\varrho)s(\varrho) + s^3(\varrho).
\end{aligned}$$

In order to satisfy the (6) we must have the variables $\mathcal{C}(\varrho) = 0$ and $\mathcal{D}(\varrho) = 0$, then calculating $\mathcal{C}$ and $\mathcal{D}$ with regard to ϱ.

$$\frac{d\mathcal{A}}{d\varrho} = \varsigma_1(\varrho)r^{'}(\varrho) - \varsigma_2(\varrho)s^{'}(\varrho) + \varsigma_3(\varrho) = 0, \tag{7}$$

$$\frac{d\mathcal{B}}{d\varrho} = \varsigma_2(\varrho)r^{'}(\varrho) + \varsigma_1(\varrho)s^{'}(\varrho) + \varsigma_4(\varrho) = 0, \tag{8}$$

where, $\varsigma_1 = 3r^2(\varrho) + 2r(\varrho)\mathcal{A}_1(\varrho) - 3s^2(\varrho) + \mathcal{A}_2(\varrho), \varsigma_2 = 6r(\varrho)s(\varrho) + 2s(\varrho)a_1(\varrho)$, $\varsigma_3 = r^2(\varrho)\mathcal{A}_1^{'}(\varrho) + s^2(\varrho)\mathcal{A}_1^{'}(\varrho) + \mathcal{A}_2^{'}(\varrho)r(\varrho), \varsigma_4 = \mathcal{A}_2^{'}(\varrho)s(\varrho) + 2r(\varrho)s(\varrho)\mathcal{A}_1^{'}(\varrho)$. On multiplying (7) by $\varsigma_1(\varrho)$ and (8) by $\varsigma_2(\varrho)$ respectively

$$r(\varrho)^{'} = -\frac{\varsigma_1(\varrho)\varsigma_3(\varrho) + \varsigma_2(\varrho)\varsigma_4(\varrho)}{\varsigma_1{}^2(\varrho) + \varsigma_2{}^2(\varrho)}. \tag{9}$$

Substituting $r(\varrho) = 0$ and $s(\varrho) = \sqrt{\mathcal{A}_2(\varrho)}$ at $\varrho = \varrho^*$ on $\varsigma_1(\varrho), \varsigma_2(\varrho), \varsigma_3(\varrho)$, and $\varsigma_4(\varrho)$, we obtain $\varsigma_1(\varrho^*) = -2\mathcal{A}_2(2^*), \varsigma_2(\varrho^*) = 2\mathcal{A}_1(\varrho^*)\sqrt{\mathcal{A}_2(\varrho^*)}, : \varsigma_3(\varrho^*) = \mathcal{A}_3^{'}(\varrho^*) - \mathcal{A}_2(\varrho^*)\mathcal{A}_1^{'}(\varrho^*)$, $\varsigma_4(\varrho^*) = \mathcal{A}_2^{'}(\varrho^*)\sqrt{\mathcal{A}_2\varrho^*}$. The Equation (9), implies

$$r^{'}(\varrho^*) = \frac{\mathcal{A}_3^{'}(\varrho^*) - (\mathcal{A}_1(\varrho^*\mathcal{A}_2(\varrho^*)))}{2(\mathcal{A}_2(\varrho^*) + \mathcal{A}_1^2(\varrho^*))}, \tag{10}$$

if $\mathcal{A}_3'(\varrho^*) - (\mathcal{A}_1(\varrho^*)\mathcal{A}_2(\varrho^*))' \neq 0$ which implies that $\frac{d}{d\varrho^*}(Re(\lambda(\varrho^*)))| \neq 0$, and $\lambda_3(\varrho^*) = -\mathcal{A}_1(\varrho^*) \neq 0$. Therefore the condition $\mathcal{A}_3'(\varrho^*) - (\mathcal{A}_1(\varrho^*)\mathcal{A}_2(\varrho^*))' \neq 0$ It has been guaranteed that the transversality criterion is satisfied, hence the model (4) has attained the Hopf-bifurcation at $\varrho = \varrho^*$. □

6. Numerical Simulations

In this section, several numerical experiments on the system (4) are carried out to verify the mathematical findings. The rate of fear ϱ is used as a control parameter. For the specified fixed parameter values, the numerical simulation is carried out using the MATLAB/MATHEMATICA software packages. With Runge-Kutta's numerical scheme.Here $r = 0.2, \alpha = 0.3, \theta = 0.25, d = 0.1, \delta = 0.1, \zeta = 0.15, \eta = 0.15, \varrho = variable.$

Bifurcation of Fear ϱ

If $\varrho = 0.3$, then the model (4) is asymptotically stable about the positive point of equilibria $E^*(0.52861, 0.0917829, 0.204774)$ and other parameter values are the same. Now, we increased the value of the bifurcation parameter, $\varrho = 0.6$, and the model (4) lost its stability, arising a limit cycle at $E^*(0.4899, 0.0920924, 0.220149)$. The model (4) then meets the transversality criteria for $(Re(\lambda(\varrho)))|_{\varrho=\varrho^*} = 0.002185 \neq 0$. The Figure 1 shows the behavioural shifts of the system (4) at impact of fear, $\varrho = 0.6$.

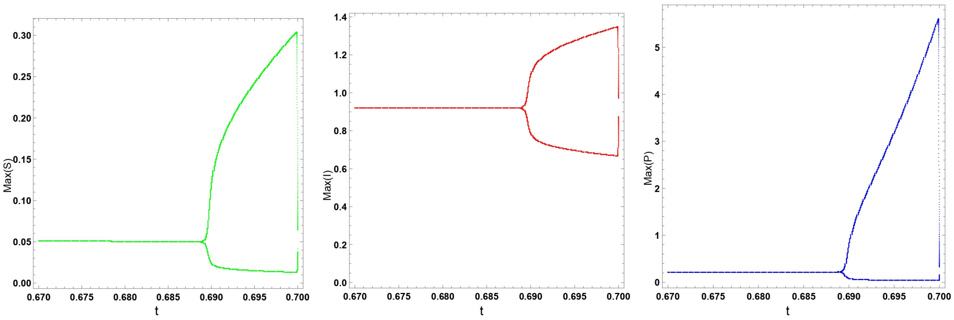

Figure 1. The dynamical change of the model (4) at $\varrho = 0.6$.

7. Conclusions

We researched an eco-epidemiological system that included infection in the population density of prey and fear in the susceptible prey population density as a result of predator attacks on susceptible and diseased prey. In addition, each biologically possible point of equilibrium can be represented (4). Furthermore, we investigated the suggested model's local stability (4) and observed the occurrence of Hopf-bifurcation, and we determined that modifying the cost of fear ϱ has an instantaneous effect on the model's stability (4). As a result, Hopf-bifurcation constrained the developed analytical arguments around the E^* simulation findings. In the proposed models, we deduce that the existence of dread has a higher impact on stability shifts via the Hopf bifurcation.

Author Contributions:

Funding:

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflict of interest.

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