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The Odd Beta Prime Inverted Kumaraswamy Distribution with Application to COVID-19 Mortality Rate in Italy †

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Abstract: Inverted distributions, also known as inverse distributions, are essential statistical models for analyzing real life data in biomedical sciences, engineering, and other fields. In this paper, we use the odd beta prime-G family and the inverted Kumaraswamy distribution to introduce a new inverted distribution called the odd beta prime inverted Kumaraswamy. The new distribution exhibits right-skewed, J-shaped densities and features increasing-constant, concave-convex, and bathtub hazard functions. Some of its statistical properties are explored. The parameters are estimated via the maximum likelihood method. The empirical importance of the new model is proved through its application to COVID-19 mortality data from Italy. Numerical results demonstrated that the proposed model outperforms its competitors. We hope that this proposed distribution can be considered as a viable alternative to some well-established distributions for modeling real life data across various application areas.

Keywords: odd beta prime-G family; Kumaraswamy distribution; inverted Kumaraswamy distribution; quantile function; infectious disease; COVID-19; mortality rate

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1. Introduction

As data sets become increasingly complex and diverse, researchers attempt to develop more statistical models that provide reliable and accurate prediction of the underlying processes [1]. Inverted distributions, also known as inverse distributions, are versatile statistical models that have a wide range of applications in a variety of practical disciplines, including the survival analysis, reliability theory, environmental studies, finance literature, econometrics, life testing problems, medical research, survey sampling, engineering sciences, and biological sciences. Their flexibility makes them valuable in modeling and analyzing various real life phenomena and making informed decisions in research and practical applications. The inverted distributions are sometimes very useful to explore additional properties of phenomena that cannot be explored using non-inverted distributions [2].

Several researchers have focused on studying inverted distributions and exploring their applications in various fields. For instance, reference [3] introduced the inverse Weibull distribution, reference [4] initiated the inverted gamma distribution, reference [5] proposed the inverse Rayleigh distribution, reference [6] studied the inverted Burr XII distribution, reference [7] pioneered the inverted Pareto I distribution, reference [8] defined the inverted Pareto II distribution, reference [9] established the inverted exponential distribution, reference [10] offered the inverse Nakagami-m distribution, reference [11] constructed the inverse Lindley distribution, reference [12] presented the inverted

Kumaraswamy distribution, reference [13] investigated the inverse power Lomax distribution, reference [14] developed the inverted Nadarajah–Haghighi distribution, reference [15] created the inverted Topp-Leone distribution, and reference [16] suggested the inverted Gompertz-Fréchet distribution.

In 1980, Kumaraswamy presented a distribution that is similar to the beta distribution but has certain significant advantages, including an inverted closed-form cumulative distribution function, and it provides simple quantile and distribution functions without the need for complex mathematical operations. This distribution can be used to model analyze a wide range of natural phenomena with lower and upper bounds, including such as the height of individuals, scores obtained on a test, atmospheric temperatures and hydrological data such as daily rain fall and daily stream flow [17]. For more details, we refer the interested readers to the following references: [18-22].

The inverted Kumaraswamy distribution was constructed via the Kumaraswamy (K) distribution using the transformation $(T) = \frac{1}{Y} - 1,$ when Y has a K distribution with probability density given as follows:

$$f(y;a,b) = aby^{a-1}\left[1 - y^{a}\right]^{b-1}; \quad 0 < y < 1, \ a,b > 0. \tag{1}$$

Thus, the distribution of T is called the inverse or inverted Kumaraswamy (IK) distribution and its domain is $(0,\infty)$. Here, we adopted the IK distribution introduced by [12] as a baseline distribution, which has the following cumulative distribution function (CDF) and probability density function (PDF), respectively:

$$F(t;a,b) = \left[1 - (1+t)^{-a}\right]^{b}; \quad t > 0, \ a,b > 0, \tag{2}$$

$$f(t;a,b) = ab(1+t)^{-a-1}\left[1 - (1+t)^{-a}\right]^{b-1}; \quad t > 0, \ a,b > 0, \tag{3}$$

where a and b are the shape parameters.

In the past few years, there has been significant interest in extending conventional distribution models to better capture real life data by employing a generalized class of distributions. These include the extended Gumbel-Weibull distribution by [23], the new flexible exponentiated Weibull distribution by [24], the generalized odd beta prime family of distributions by [25], the new extended Topp–Leone exponential distribution by [26], the log-Topp-Leone distribution by [27], the Marshall–Olkin extended Gumbel type-II distribution by [28], the McDonald generalized power Weibull distribution by [29], the exponentiated odd Lomax exponential distribution by [30], the Maxwell-Weibull distribution by [31], the Maxwell-exponential distribution by [32], the odd-F-Weibull distribution by [33], and many others. Recently, reference [34] developed a new family of distribution referred to as the odd beta prime-G (OBP-G) family. The CDF and PDF of the OBP-G family are respectively given by:

$$F(t;c,d,\delta) = \frac{B_{\frac{Q(t,\delta)}{1-Q(t,\delta)}}(c,d)}{B(c,d)}; \quad t > 0, \ c,d > 0, \tag{4}$$

and

$$f(t;c,d,\delta) = \frac{q(t,\delta)}{B(c,d)\{1 - Q(t,\delta)\}^{2}} \frac{\left\{\frac{Q(t,\delta)}{1-Q(t,\delta)}\right\}^{c-1}}{\left\{1 + \left(\frac{Q(t,\delta)}{1-Q(t,\delta)}\right)\right\}^{c+d}}; \quad t > 0, \ c,d > 0, \tag{5}$$

where c and d are the shape parameters, $Q(t,\delta)$ and $q(t,\delta)$ are the CDF and PDF of the baseline distribution with parameter δ, respectively. The OBP-G class has been employed to extend several baseline distributions, resulting in new compound distributions with different properties and applications. For instance, reference [35] proposed the OBP-logistic distribution, while reference [36] developed the OBP-exponential for analyzing hydrological and engineering data, reference [37] proposed the OBP-Fréchet and applied it to groundwater data, reference [38] created the OBP-Burr X and applied it to model petroleum rock samples, and more.

This study aims to suggest a new extension of the IK distribution by utilizing the OBP-G class, which is named as the odd beta prime-inverted Kumaraswamy (OBPIK) distribution. The proposed OBPIK distribution exhibits greater flexibility in modeling data sets with a long right tail compared with other commonly used distributions. As a result, the OBPIK can be efficiently used for long term reliability estimates, producing accurate predictions of extreme values occurring in the right tail of the distribution compared with other distributions.

COVID-19 is a new viral disease caused by the severe acute respiratory syndrome coronavirus-2 (SARS-CoV-2) that generated a global epidemic. Several mathematical and statistical models have been proposed to explain the path of the pandemic [39, 40]. It is important to point out that the characteristics of the pandemic data can fluctuate, making it unable to fit classical probability distributions in all cases. As a result, we have developed the OBPIK distribution to model the mortality rate of this infectious disease in Italy.

The motivation and justification for introducing the OBPIK distribution are as follows:

(i) to improve the general performance of the classical IK distribution, which can handle right-skewed and heavy-tailed data sets when compared to other competitive models;
(ii) to develop a model with different shapes, such as right-skewed and reversed-J shape;
(iii) to introduce a new model with various hazard functions that can capture increasing, bathtub, and concave-convex shapes; and
(iv) to consistently offer superior fit in comparison to well-established, generated distributions for the same baseline distribution.

For these reasons, we proposed the OBPIK distribution, made up of the combination of the odd beta prime family of distributions proposed by [34] and the inverted Kumaraswamy distribution

This paper is outlined as follows: Section 2 contains the development of the OBPIK distribution. Section 3 provides some of its basic statistical properties. Section 4 highlights the method of parameter estimation. Section 5 provides the numerical application of the new model. Section 6 gives the concluding remarks.

2. The Odd Beta Prime Inverted Kumaraswamy Distribution

The odd beta prime inverted Kumaraswamy (OBPIK) model is generated by introducing two additional shape parameters from the OBP-G family. The CDF of the OBPIK model is obtained by inserting (2) into (4) as provided via:

$$F\left(t;a,b,c,d\right)=\frac{B_{\frac{\left[1-(1+t)^{-a}\right]^{b}}{\left(1-\left[1-(1+t)^{-a}\right]^{b}\right)}}\left(c,d\right)}{B\left(c,d\right)};\quad t>0, \tag{6}$$

where $a,b,c,d > 0$ are the shape parameters. The corresponding PDF is derived by inserting (3) into (5) as provided via:

$$f(t;a,b,c,d) = \frac{ab(1+t)^{-a-1}\left[1-(1+t)^{-a}\right]^{bc-1}}{B(c,d)\left\{1-\left[1-(1+t)^{-a}\right]^{b}\right\}^{1-d}}; \quad t>0. \tag{7}$$

For simplicity the parameters on the CDF and PDF are omitted by writing $F(t;a,b,c,d)=F(t)$ and $f(t;a,b,c,d)=f(t)$, respectively. The PDF plots of the OBPIK model with various parameter combinations are displayed in Figure 1. The PDF of the OPBIK can exhibit either (a) right-skewed or (b) reversed-J shapes.

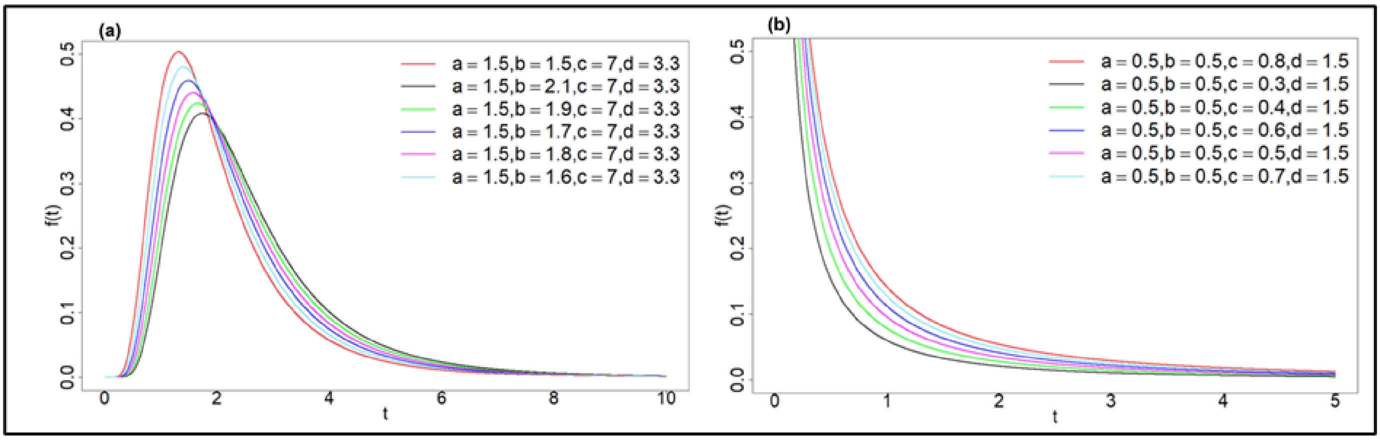

Figure 1. The PDFs plots of the OBPIK model with various parameter values.

The survival function of the OBPIK model is obtained from (6) as provided via:

$$S(t) = 1 - \frac{B_{\frac{\left[1-(1+t)^{-a}\right]^{b}}{\left(1-\left[1-(1+t)^{-a}\right]^{b}\right)}}(c,d)}{B(c,d)}; \quad t>0. \tag{8}$$

The hazard function (HF) of the OBPIK model is derived from (6) and (7) as:

$$h(t) = \frac{\dfrac{ab(1+t)^{-a-1}\left[1-(1+t)^{-a}\right]^{bc-1}}{B(c,d)\left\{1-\left[1-(1+t)^{-a}\right]^{b}\right\}^{1-d}}}{1-\dfrac{B_{\frac{\left[1-(1+t)^{-a}\right]^{b}}{\left(1-\left[1-(1+t)^{-a}\right]^{b}\right)}}(c,d)}{B(c,d)}}; \quad t>0. \tag{9}$$

The HF plots of the OBPIK model with various parameter combinations of are shown in Figure 2. These plots indicate that the HF of the OPBIK model can exhibit either (a) increasing gradually to the peak then constant or (b) Concave-convex and bathtub shapes.

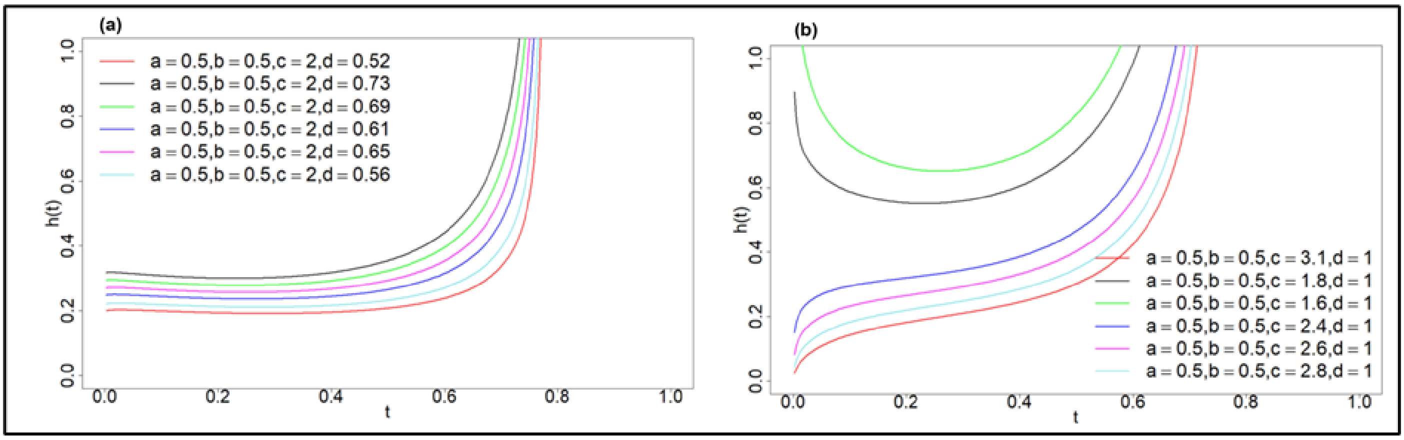

Figure 2. The HFs plots of the OBPIK model with various parameter values.

3. Properties of the Odd Beta Prime Inverted Kumaraswamy Distribution

Some of the basic properties of the OBPIK model derived in this section include the moments, moment generating function, and quantile function.

3.1. Moments

By the definition of moments, the moments of the OBPIK distribution can be given via:

$$E\left(t^r\right)=\int_{-\infty}^{\infty} t^r f\left(t\right)dt, \tag{10}$$

where $f\left(t\right)$ is the PDF of the OBPIK defined in (7). By inserting (7) into (10), we get:

$$E\left(t^r\right)=\frac{ab}{B\left(c,d\right)}\int_0^{\infty} t^r \frac{\left(1+t\right)^{-a-1}\left[1-\left(1+t\right)^{-a}\right]^{bc-1}}{\left\{1-\left[1-\left(1+t\right)^{-a}\right]^{b}\right\}^{1-d}}dt. \tag{11}$$

After algebra, we obtained the moments of the OBPIK distribution given by:

$$E\left(t^r\right)=\frac{ab}{B\left(c,d\right)}\sum_{i,j=0}^{\infty}\psi_{i,j}\, B\left(a(1+j)-r\ ,\ r+1\right), \tag{12}$$

where $\psi_{i,j}=\dfrac{(-1)^j\,\Gamma(1-d+i)}{i!\,\Gamma(1-d)}\dbinom{b(c+i)-1}{j}$.

3.2. Moment Generating Function

The moment generating function (MGF) of the OBPIK model is provided via:

$$M_T\left(x\right)=E\left(e^{tx}\right)=\int_{-\infty}^{\infty} e^{tx} f\left(x\right)dx=\int_{-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(tx)^k}{k!} f\left(x\right)dx=\sum_{k=0}^{\infty}\frac{x^k}{k!}E\left(t^k\right). \tag{13}$$

By setting $r=k$ in (12) and insert it in (13), we get the MGF of the OBPIK given by:

$$M_T(x) = \frac{ab}{B(c,d)} \sum_{i,j,k=0}^{\infty} \psi_{i,j} \frac{x^k}{k!} B\left(a(1+j)-k\ ,\ k+1\right). \tag{14}$$

3.3. Quantile Function

The quantile function (QF) of the OBPIK distribution is formulated by inverting (6) as follows:

$$T = \left\{1 - K^{\frac{1}{b}}\right\}^{-\frac{1}{a}} - 1, \tag{15}$$

where $K = \frac{I^{-1}(u;c,d)}{1+I^{-1}(u;c,d)}$ and $I^{-1}(u;c,d)$ is the inverted CDF of (6).

4. Estimation of Parameters

This section presents the estimation of parameters of the OBPIK distribution using the maximum likelihood method. Let $T_1, T_2, \ldots, T_n$ be a random variable of sample size n from the OBPIK model with parameters $a, b, c,$ and d, then its likelihood function is derived from (7) as provided via:

$$L = \left\{\frac{ab}{B(c,d)}\right\}^n \prod_{i=1}^{n} \frac{(1+t_i)^{-a-1}\left[1-(1+t_i)^{-a}\right]^{bc-1}}{\left\{1-\left[1-(1+t_i)^{-a}\right]^b\right\}^{1-d}}. \tag{16}$$

The log-likelihood function of (16) presented by l is provided via:

$$\begin{aligned} \mathrm{l} = n\log\left\{\frac{ab}{B(c,d)}\right\} + (-a-1)\sum_{i=1}^{n}\log(1+t_i) + (bc-1)\sum_{i=1}^{n}\log\left[1-(1+t_i)^{-a}\right] \\ -(1-d)\sum_{i=1}^{n}\log\left\{1-\left[1-(1+t_i)^{-a}\right]^b\right\}. \end{aligned} \tag{17}$$

Obviously, software likes R or MATLAB can be used to obtain these solutions.

5. Numerical Illustration to COVID-19 Mortality Rate

Here, the application of the OBPIK distribution is validated by using COVID-19 data. These data represent the COVID-19 mortality rates of Italy recorded for a period of 111 days from 1 April to 20 July 2020. The data can be found in [41]. These data set are presented as follows: 0.2070, 0.1520, 0.1628, 0.1666, 0.1417, 0.1221, 0.1767, 0.1987, 0.1408, 0.1456, 0.1443, 0.1319, 0.1053, 0.1789, 0.2032, 0.2167, 0.1387, 0.1646, 0.1375, 0.1421, 0.2012, 0.1957, 0.1297, 0.1754, 0.1390, 0.1761, 0.1119, 0.1915, 0.1827, 0.1548, 0.1522, 0.1369, 0.2495, 0.1253, 0.1597, 0.2195, 0.2555, 0.1956, 0.1831, 0.1791, 0.2057, 0.2406, 0.1227, 0.2196, 0.2641, 0.3067, 0.1749, 0.2148, 0.2195, 0.1993, 0.2421, 0.2430, 0.1994, 0.1779, 0.0942, 0.3067, 0.1965, 0.2003, 0.1180, 0.1686, 0.2668, 0.2113, 0.3371, 0.1730, 0.2212, 0.4972, 0.1641, 0.2667, 0.2690, 0.2321, 0.2792, 0.3515, 0.1398, 0.3436, 0.2254, 0.1302, 0.0864, 0.1619, 0.1311, 0.1994, 0.3176, 0.1856, 0.1071, 0.1041, 0.1593, 0.0537, 0.1149, 0.1176, 0.0457, 0.1264, 0.0476, 0.1620, 0.1154, 0.1493, 0.0673, 0.0894, 0.0365, 0.0385, 0.2190, 0.0777, 0.0561, 0.0435, 0.0372, 0.0385, 0.0769, 0.1491, 0.0802, 0.0870, 0.0476, 0.0562, 0.0138.

Table 1 shows the descriptive statistics of these data. It is obvious that the data has a right tail and platykurtic. The histogram in Figure 3 confirms that the data has a right tail, and the extreme values are spotted in the box plot. This validates that the shape of the density function of the proposed OBPIK model provided in Figure 1 is appropriate for modeling this type of data.

Table 1. Descriptive statistics for COVID-19 mortality rate in Italy.

Statistic	Min	Q1	Q3	Median	Mean	Max	Std. dev	Skewness	Kurtosis
Value	0.0138	0.1201	0.2064	0.1628	0.1668	0.4972	0.0788	0.7624	1.8129

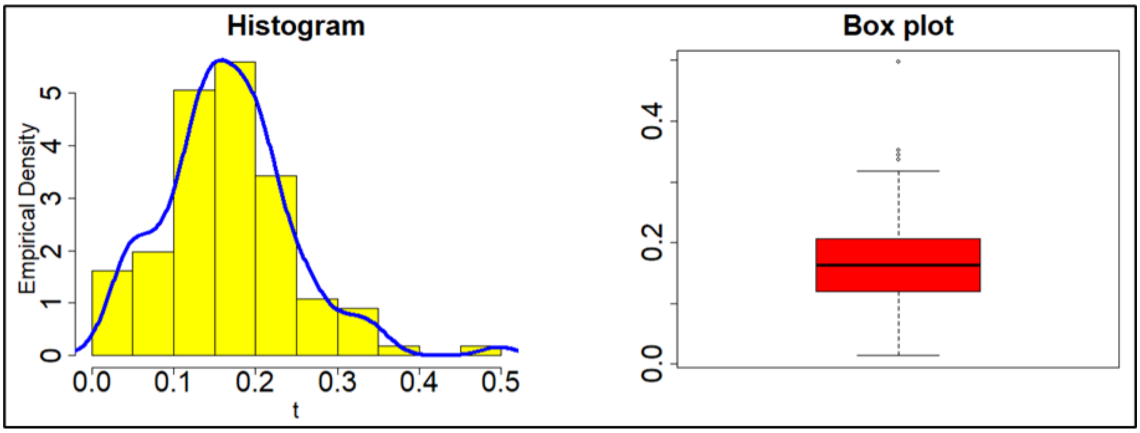

Figure 3. Histogram and box plot for COVID-19 mortality rate in Italy.

To verify the performance of the OBPIK model, we compare its fit with that of its related models, such as the inverted Kumaraswamy (IK), the Topp Leone-generalized inverted Kumaraswamy (TLGIK), and the Marshall-Olkin extended inverted Kumaraswamy (MOEIK). We use the values of the negative log-likelihood function $\left(-\hat{l}\right)$, Akaike information criterion (AIC), corrected AIC (CAIC), Bayesian information criterion (BIC), and Hannan–Quinn information criterion (HQIC) to select the best fitted model for these data. The best fitting model is the one that has the maximum value of $-\hat{l}$ and provides the lowest values of the aforementioned fitted criteria. The maximum likelihood estimates (MLEs) and the fitted measures for the parameters of all competing models are given in Table 2.

Table 2 shows that the OBPIK model exhibits the lowest fitted measures compared to all other fitted models. Therefore, it can be selected as the best model for analyzing the COVID-19 mortality rate in Italy. Figure 4(a) displays the plots of the fitted densities for the COVID-19 data. Figure 4(b) shows the plots of empirical and fitted CDFs for the same COVID-19 data. These figures confirmed the results presented in Table 2.

Table 2. MLEs of each distribution for COVID-19 mortality rate in Italy.

Model	Estimates					Fitted Measures			
	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{d}$	$-\hat{l}$	AIC	CAIC	BIC	HQIC
OBPIK	1.8061	0.9866	1.4746	1.9989	20.7935	-33.5870	-32.8027	-25.4856	-30.4461
IK	1.9465	1.1201	---	---	1.9899	0.0200	0.2464	4.0707	1.5904
TLGIK	1.5382	0.8963	1.1591	1.1737	20.6626	-33.3252	-32.5409	-25.2238	-30.1843
MOEIK	1.2582	1.4474	0.2106	---	16.2904	-26.5809	-26.1193	-20.5048	-24.2252

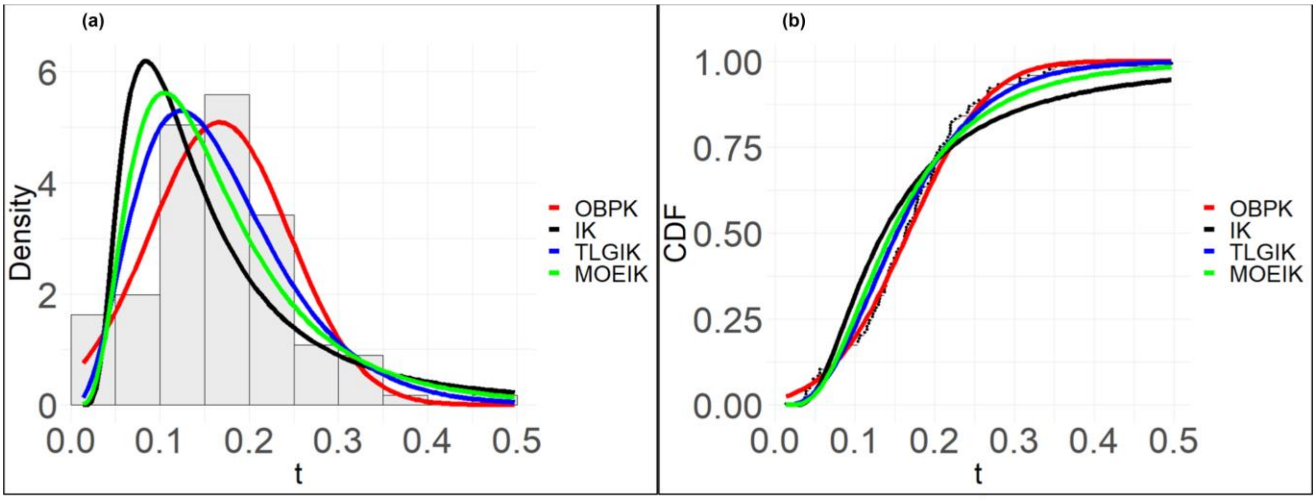

Figure 4. Fitted PDFs and CDFs of the competing models for COVID-19 mortality rate in Italy.

6. Concluding Remarks

In this paper, we propose a new member of the inverted distribution called the odd beta prime inverted Kumaraswamy distribution. Some basic statistical properties of the new model, including the moment, moment generating function, and quantile function, are viewed. The maximum likelihood estimators of the model are derived. To demonstrate the importance and applicability of the new model, we apply it to the COVID-19 mortality rate in Italy. Numerical results show that the proposed model outperforms other comparable models. We hope that the new distribution can be considered a good alternative to some well-established distributions for real life data modeling in various areas of application.

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