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Fuzzy Triangular Finite Elements solution for solving the Nonlinear Boussinesq Equation

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INTRODUCTION & PROBLEM HYPOTHESIS

Fuzzy unsteady flow in a semi-infinite unconfined aquifer bordering a lake

Problem considerations

- **Crisp** initially and boundary conditions
- **Fuzzy** hydraulic conductivity (K) and specific yield (S)

Problem approach

[1, 2] Theory of fuzzy partial generalized differentiabilty

Application to the problem

The theory has applied translating the fuzzy problem to a **system** of second-order crisp boundary value problems, called **corresponding systems.**

The main goal were threefold

- to develop **a fuzzy FEM** numerical scheme based on **triangular elements** tailored to the examined physical problem.
- to establish the **range** in which the semi-analytical solution **approximates** the **new** triangular fuzzy FEM solution, and to perform a **comparative** analysis with both a **semi-analytical** [3] and previously developed **orthogonal** fuzzy FEM solutions [4].
- to **provide** engineers and water resource planners with **improved knowledge** of hydraulic properties under uncertainty by using **Fuzzy** and **Possibility** theory, which serves as a **bridge** between the **Fuzzy** and the **Probability** theory.

In this work, the fuzzy solution of the nonlinear Boussinesq equation is **investigated** for an unconfined aquifer bordering a lake, while the hydraulic conductivity (K) and specific yield (S) are treated as fuzzy. **Traditional** numerical techniques, such as Finite Difference Methods, have proven useful in solving various fluid dynamics problems. However, these methods **struggle** when applied to domains with **complex** geometries and **heterogeneous** boundary conditions.
Compared to traditional numerical approaches, the **proposed fuzzy Triangular Finite Element** scheme provides a more robust and flexible framework capable of **handling** irregular geometries, highly variable hydraulic properties, and the inherent uncertainties in hydrological processes.

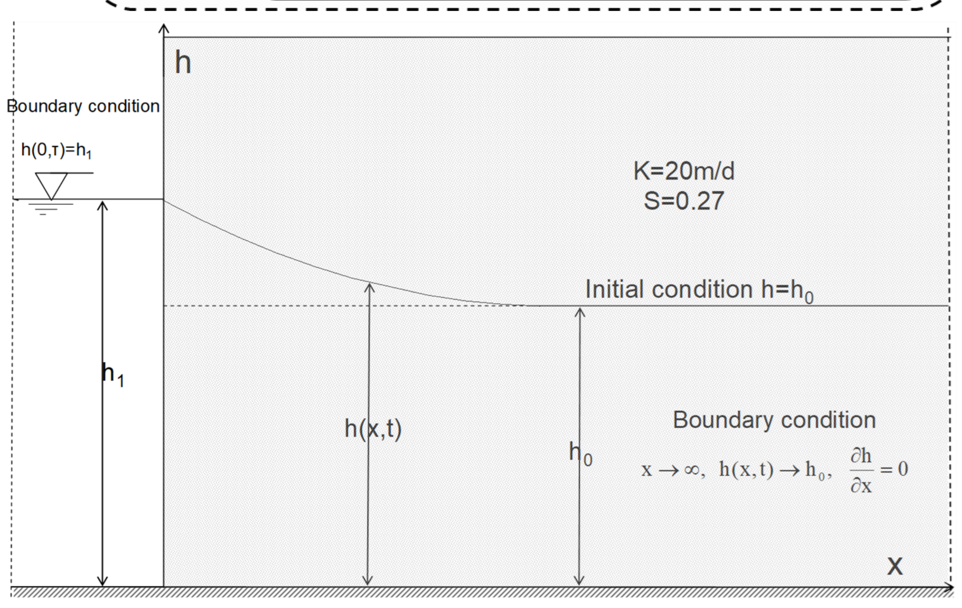

Figure 1. Definition sketch of the investigated problem

METHOD
CRISP AND FUZZY MATHEMATICAL MODELS

Crisp model with the initial and boundary conditions

$$\frac{\partial h}{\partial t}=\frac{K}{S}\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)\qquad \begin{aligned} &t=0, && h(x,0)=h_0,\\ &t>0, && h(0,t)=h_1,\quad h(\infty,t)=h_0,\quad \left.\frac{\partial h(x,t)}{\partial x}\right|_{x\to\infty}=0. \end{aligned}$$

Fuzzy model with the **new** initial and boundary conditions

$$\frac{\partial \widetilde{H}}{\partial \tau}=\frac{\partial}{\partial s}\left(2\widetilde{H}\frac{\partial \widetilde{H}}{\partial s}\right)\qquad \begin{aligned} &\tau=0, && \widetilde{H}(s,0)=\widetilde{1},\\ &\tau>0, && s=0\quad \widetilde{H}(0,\tau)=\widetilde{H}_1=\frac{\widetilde{h}_1}{\widetilde{h}_0},\\ & && s\to\infty\quad \widetilde{H}(\infty,\tau)=\widetilde{1},\quad \frac{\partial(\widetilde{H}(\infty,\tau))}{\partial s}=0. \end{aligned}$$

Solutions to the previous fuzzy problem and the boundary and initial conditions, can be find utilizing the theories of [2-4], translating the above fuzzy problem to a **system of second order of crisp boundary value problems**, hereafter called **corresponding system** for the fuzzy problem. Therefore, eight crisp BVPs systems are possible for the fuzzy problem with the same initial and boundary conditions. We have hereby restricted ourselves to the solution of the first system:

$$\frac{\partial H^-}{\partial \tau}=H^-\frac{\partial^2 H^-}{\partial \xi^2}+\left(\frac{\partial H^-}{\partial \xi}\right)^2 \quad\Big|\quad \frac{\partial H^+}{\partial \tau}=H^+\frac{\partial^2 H^+}{\partial \xi^2}+\left(\frac{\partial H^+}{\partial \xi}\right)^2$$

For the solutions of the above system the dimensional form is constructed as follows:

$$\frac{\partial h^-}{\partial t}=\left(\frac{K}{S}\right)^-\frac{\partial}{\partial x}\left(h^-\frac{\partial h^-}{\partial x}\right)$$

In which parameters K and S considered fuzzy based on the following graphs (Figures 1-3).

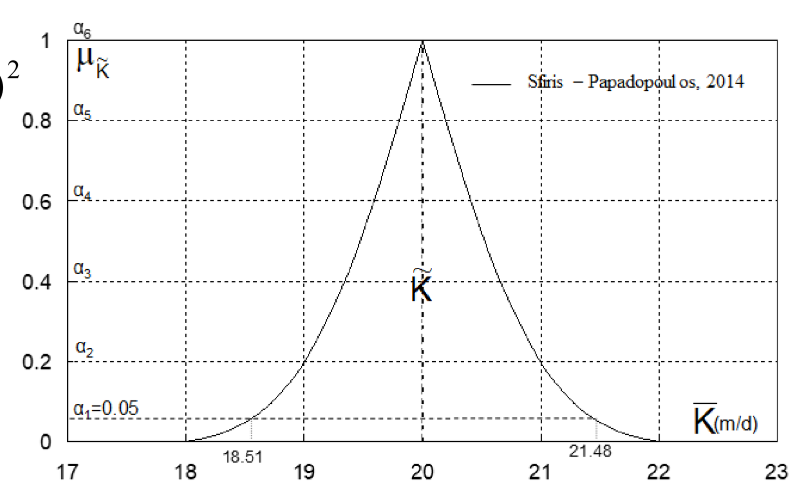

Figure 2. Fuzzy estimator K

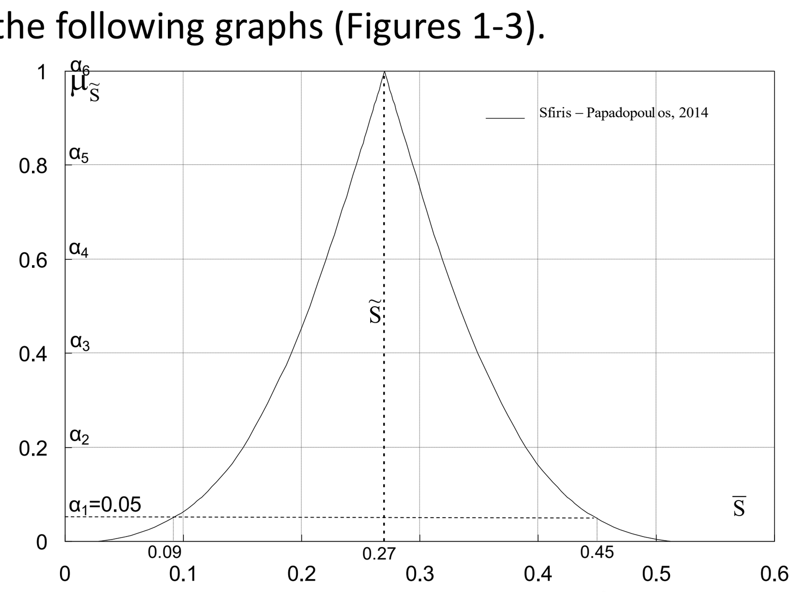

Figure 3. Fuzzy estimator of S

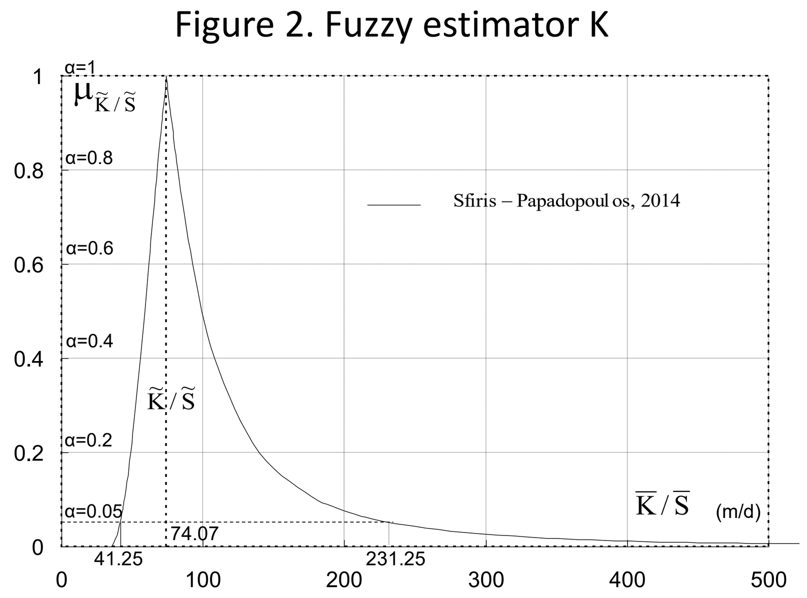

Figure 4. Fuzzy estimator of K/S

RESULTS & DISCUSSION

Construction of the application

- Efficiency **Test** of the proposed fuzzy triangular finite element solution
- Selected soil **sample** (coarse sand)
- Hydraulic parameters
  - **K** mean value **20m/d** and std. **5m/d**
  - **S** mean value **0.27** and std. **0.6**
- **Magnitube** of the measurements is equal to **n=40** and follow a **normal** distribution

Hypothesis

- The parameters **K** and **S** are considered **fuzzy**
  - The function h is **fuzzy**
  - The non dimensional function H is **fuzzy**
- In this case
  - **Initial water level** $h_0 = 4m$
  - **Abruptly raised to** $h_1 = 6m$

In the following **graphs** (Figures 5-7), it is evident that the **fuzzy triangular FEM** results are in **close agreement** with those obtained from the **fuzzy orthogonal FEM**, and **both** show **satisfactory consistency** with the **semi-analytical** solution of Lockington.

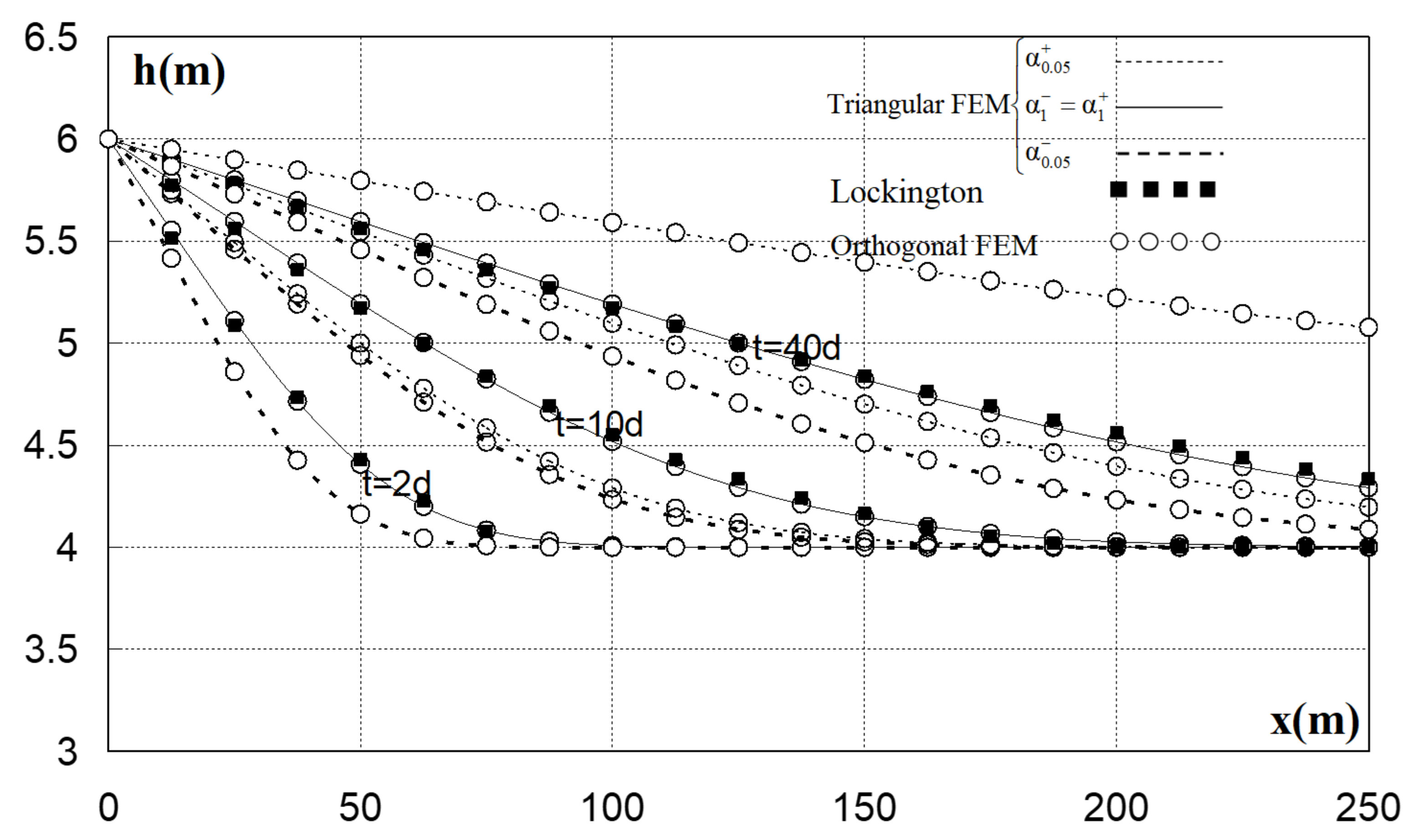

Figure 5. Ground water levels h vs. x for different times

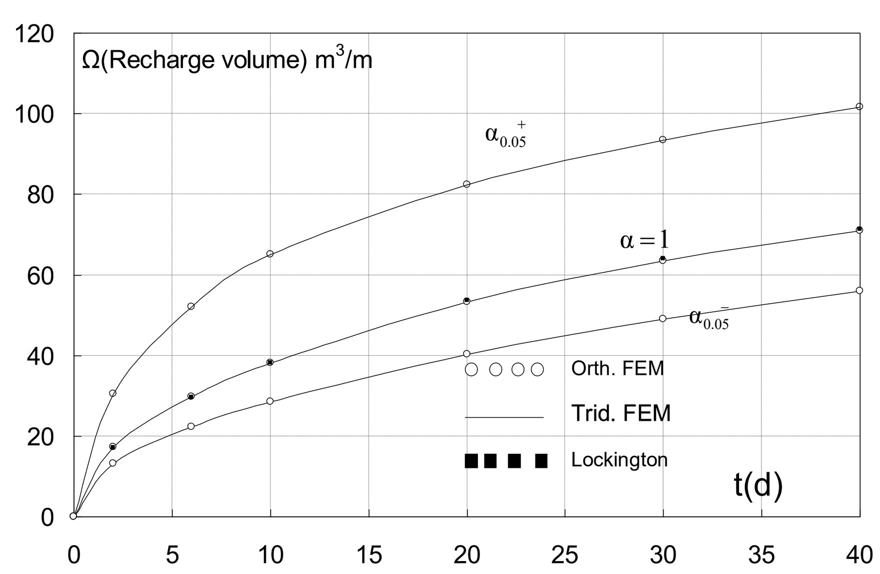

Figure 6. Water recharge volume $\Omega\ (m^3/m)$ vs. $t(d)$

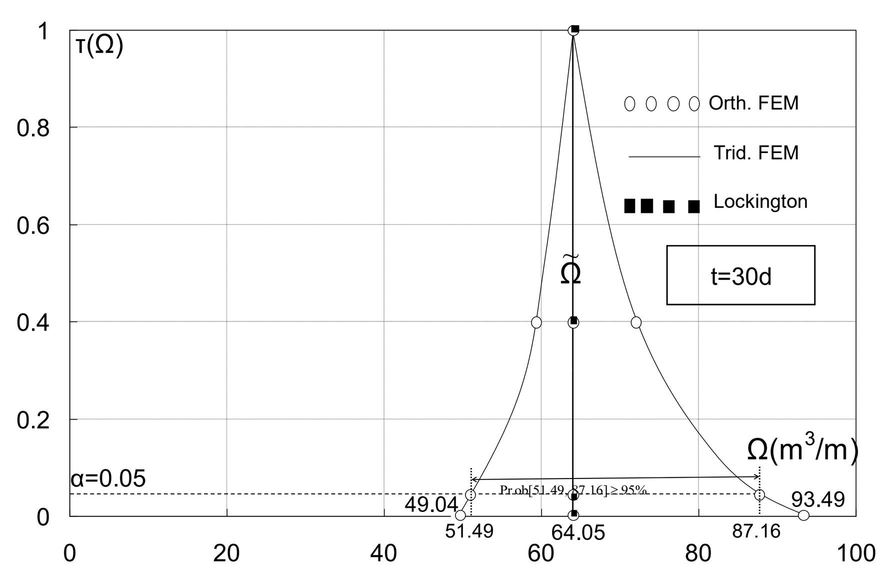

Figure 7. Fuzzy estimator of water outflow $t = 30d$

CONCLUSION

- **Compared** to traditional numerical approaches, the **proposed fuzzy scheme** provides a more robust and flexible framework capable of **handling** irregular geometries, highly variable hydraulic properties, and the inherent uncertainties in hydrological processes.
- The **integration** of triangular fuzzy logic with FEM, grounded in the **generalized Hukuhara derivative** (gH-derivative) theory for partial differential equations, represents a **substantial** theoretical innovation that extends existing fuzzy calculus to practical hydraulic modeling
- Our **comparative** analysis shows that the triangular fuzzy FEM achieves **excellent agreement** with **both** the **orthogonal** fuzzy FEM and the **Lockington** semi analytical solution. This confirms the method's **high reliability** and **predictive accuracy**.
- Overall, the benefit gained through the application of this methodology is substantial, especially for engineers involved in the **design** and **construction** of hydraulic projects.

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