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Doubly Truncated Generalized Entropy

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Abstract: Recently, the concept of generalized entropy has been proposed in the literature of information theory. In the present paper, we introduce and study the notion of generalized entropy in the interval (t_1, t_2) as uncertainty measure. It is shown that the suggested information measure uniquely determines the distribution function. Also, its properties has been studied. Some results have been obtained and some distributions such as uniform, exponential, Pareto, power series and finite range have been characterized by doubly truncated (interval) generalized entropy. Further, we describe a few orders based on this entropy and show its properties.

Keywords: life distributions, generalized entropy, uncertainty measure, doubly truncated (interval) distribution

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1. Introduction

In survival studies and life testing, information about the lifetime between two time points is available. In other words, event time of individuals which lies within a specific time interval are only observed. Thus, the analyzer cannot have access to the information about the subjects outside of this interval. For example, final products are often subject to selection checkup before being sent to the customer. The usual practice is that if a product's performance falls within certain tolerance limits, it is refereed compatible and sent to the customer. If it fails, a product is rejected and thus revoked. In this case, the actual distribution to the customer is called doubly (interval) truncated.

Nowadays, uncertainty measures has earned a great deal of authors attention. Shannon [16] was the first one who introduced entropy, known as Shannon's entropy, into information theory. For an absolutely continuous nonnegative random variable X having probability density function f, Shannon's entropy is defined as

$$H(X) = -\int_0^\infty f(x)\log f(x)\,dx = -E(\log f(X)). \tag{1}$$

It measures the expected uncertainty contained in probability density function about the predictability of an outcome of X. There are several generalizations of (1). Khinchin [9] generalized (1) and defined measure as

$$H^\phi(X) = -\int_0^\infty f(x)\,\phi(f(x))\,dx, \tag{2}$$

where ϕ is a convex function such that $\phi(1) = 0$. By choosing two particular ϕ, (2) can be rewritten as

$$H_1^\beta(X) = \frac{1}{\beta - 1}\left(1 - \int f^\beta(x)\,dx\right) \tag{3}$$

and

$$H_2^\beta(X) = \frac{1}{1 - \beta}\log\left(\int f^\beta(x)\,dx\right) \tag{4}$$

for some fixed $\beta > 0$ and $\beta \geq 1$. When $\beta \to 1$ in (3) or (4), then they tend to (1). For some distributions, $H(X)$ may be negative but one can find nonnegative $H_1^\beta(X)$ and $H_2^\beta(X)$ by choosing appropriate value of β.

When a unit studied that survived up to an age t, the Shannon's entropy is not suitable for measuring the uncertainty. So the notion of residual and past uncertainty has been introduced. Ebrahimi [6], instead of (1) defined

$$H(X,t) = -\int_t^\infty \frac{f(x)}{\overline{F}(t)}\log\frac{f(x)}{\overline{F}(t)}dx, \tag{5}$$

where $\overline{F}_X(t)$ be the survival function of the X. It is well known from (5) that units which exhibit less uncertainty in life times are more reliable and hence measure (5) has much relevance in characterizing, ordering and classifying life distributions according to its behavior. See for more details Asadi and Ebrahimi [2], Blezunce et al. [3], Ebrahimi and Pellerey [7] and Nair and Rajesh [13]. In the same spirit, Nanda and Paul [14] have extended (3) and (4) for a unit surviving up to age t as

$$H_1^\beta(X,t) = \frac{1}{\beta - 1}\left(1 - \int_t^\infty \left(\frac{f(x)}{\overline{F}(t)}\right)^\beta dx\right) \tag{6}$$

and

$$H_2^\beta(X,t) = \frac{1}{1 - \beta}\log\left(\int_t^\infty \left(\frac{f(x)}{\overline{F}(t)}\right)^\beta dx\right) \tag{7}$$

respectively. It can be noted that when $\beta \to 1$ in (6) or (7), then they tend to (5). In some practical situations, uncertainty is related to past life time rather than future. As an example, one can be find past uncertainty of a unit that failed at time t. The past entropy over $(0,t)$ of random life time X have been defined by Di Crescenzo and Longobardi [5] as

$$\overline{H}(X,t) = -\int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} dx, \tag{8}$$

where $F_X(t)$ be the distribution function of X. Gupta and Nanda [8] have defined generalized past entropies given by

$$\overline{H}_1^{\beta}(X,t) = \frac{1}{\beta - 1}\left(1 - \int_0^t \left(\frac{f(x)}{F(t)}\right)^{\beta} dx\right) \tag{9}$$

and

$$\overline{H}_2^{\beta}(X,t) = \frac{1}{1-\beta} \log\left(\int_0^t \left(\frac{f(x)}{F(t)}\right)^{\beta} dx\right) \tag{10}$$

respectively. As $\beta \to 1$ in (9) or (10), then they reduce to (8).

In some situations, information between two points is considered. Therefore statistical measures in information theory under condition of doubly truncated random variables must be studied. A dynamic uncertainty measure for two sided truncated random variables has been discussed by Sunoj et al. [17], Misagh and Yari [11] and Misagh and Yari [12] as an extension of Shannon entropy. They consider the notion of interval entropy of random life time X in the interval (t_1, t_2) as an uncertainty measure contained in $(X|t_1 < X < t_2)$ as

$$IH(X,t_1,t_2) = -\int_{t_1}^{t_2} \frac{f(x)}{F(t_2) - F(t_1)} \log \frac{f(x)}{F(t_2) - F(t_1)} dx. \tag{11}$$

In this paper, an effort is made to develop some new characterizations to certain probability distributions and families of distributions using definition of doubly truncated generalized entropy which are suitable for modeling and analysis of lifetime data. This paper is arranged as follows; in section 2, as preliminaries, first and second kind of generalized interval entropies defined. Properties of these entropies obtained in section 3. In section 4, a few ordering results are shown based of entropies defined in section 2. Finally, conclusion is illustrated in last section.

2. Preliminaries

In this section, we define first and second kind of generalized interval entropies as uncertainty measures and then these definitions obtained for some distributions.

Definition 2.1.

i) The first kind of generalized interval entropy of order β for a random lifetime Y between time t_1 and t_2 is

$$IH_1^{\beta}(X,t_1,t_2) = \frac{1}{\beta-1}\left(1 - \int_0^{\infty} (f_X(y))^{\beta} dx\right), \quad \text{for } \beta \neq 1, \ \ \beta > 0 \tag{12}$$

ii) The second kind of generalized interval entropy of order β for a random lifetime Y between time t_1 and t_2 is

$$IH_2^{\beta}(X,t_1,t_2) = \frac{1}{1-\beta} \log\left(\int_0^{\infty} (f_X(y))^{\beta} dx\right), \quad \text{for } \beta \neq 1, \ \ \beta > 0 \tag{13}$$

where $f_X(y)$ is the probability density function of $Y \overset{d}{=} X|t_1 < X < t_2$ and $(t_1, t_2) \in D = \left\{(u,v) \in R^{+^2}; F(u) \leq F(v)\right\}$.

Relations (12) and (13) for some $\beta > 0$ and $\beta \neq 1$ can be rewritten as

$$\begin{aligned} IH_1^\beta(X, t_1, t_2) &= \frac{1}{\beta - 1}\left(1 - \int_{t_1}^{t_2}\left(\frac{f(x)}{F(t_2) - F(t_1)}\right)^\beta dx\right) \\ &= \frac{1}{\beta - 1}\left(1 - E\left(\frac{f(X)}{F(t_2) - F(t_1)}\right)^{\beta - 1}\right) \end{aligned} \tag{14}$$

and

$$\begin{aligned} IH_2^\beta(X, t_1, t_2) &= \frac{1}{1 - \beta}\log\left(\int_{t_1}^{t_2}\left(\frac{f(x)}{F(t_2) - F(t_1)}\right)^\beta dx\right) \\ &= \frac{1}{1 - \beta}\log E\left(\frac{f(X)}{F(t_2) - F(t_1)}\right)^{\beta - 1} \end{aligned} \tag{15}$$

respectively. Equations (14) and (15) leads to

$$(\beta - 1) IH_1^\beta(X, t_1, t_2) = 1 - (F(t_2) - F(t_1))^{-\beta}\int_{t_1}^{t_2}(f(x))^\beta dx \tag{16}$$

and

$$(1 - \beta) IH_2^\beta(X, t_1, t_2) = \log\int_{t_1}^{t_2}(f(x))^\beta dx - \beta\log(F(t_2) - F(t_1)) \tag{17}$$

respectively. When the system has the age t_1, for different values of β, $IH_i^\beta(X, t_1, t_2)$ provides the information spectrum of the systems remaining life until age t_2.

Also, we have $\underset{t_1 \to 0^+}{lim} IH_1^\beta(X, t_1, t_2) = \overline{H}_1^\beta(X, t_2)$ and $\underset{t_2 \to \infty}{lim} IH_1^\beta(X, t_1, t_2) = H_1^\beta(X, t_1)$. The same results hold for $IH_2^\beta(X, t_1, t_2)$.

In the Example 2.1, $IH_1^\beta(X, t_1, t_2)$ and $IH_2^\beta(X, t_1, t_2)$ obtained for some distributions. We first give definition of general failure rate (GFR) functions extracted from Navarro and Ruiz [15].

Definition 2.1. The GFRs of a random variable X having density function $f(x)$ and cumulative distribution function $F(x)$ are given by $h_j(t_1, t_2) = \frac{f(t_j)}{F(t_2) - F(t_1)}$, $j = 1, 2$.

Example 2.1. Let X be a random variable with

i) exponential distribution with survival function $\overline{F}(x) = e^{-\theta x}; x > 0$ then

$$IH_1^\beta(X, t_1, t_2) = \frac{1}{\beta - 1}\left(1 + \frac{1}{\theta\beta}\left[h_2^\beta(t_1, t_2) - h_1^\beta(t_1, t_2)\right]\right) \tag{18}$$

and

$$IH_2^\beta(X, t_1, t_2) = \frac{1}{1 - \beta}\log\left(-\frac{1}{\theta\beta}\left[h_2^\beta(t_1, t_2) - h_1^\beta(t_1, t_2)\right]\right) \tag{19}$$

ii) finite range distribution with survival function $\overline{F}(x) = (1 - ax)^b; 0 < x < \frac{1}{a}, b > 0, a > 0$ then

$$IH_1^\beta(X, t_1, t_2) = \frac{1}{\beta - 1}\left(1 - \frac{1}{a(\beta(b - 1) + 1)}\left[(1 - at_1) h_1^\beta(t_1, t_2) - (1 - at_2) h_2^\beta(t_1, t_2)\right]\right) \tag{20}$$

and

$$IH_2^{\beta}(X,t_1,t_2)=\frac{1}{1-\beta}\log\left(\frac{1}{a(\beta(b-1)+1)}\left[(1-at_1)h_1^{\beta}(t_1,t_2)-(1-at_2)h_2^{\beta}(t_1,t_2)\right]\right) \quad (21)$$

iii) Pareto II distribution with survival function $\overline{F}(x)=(1+px)^{-q};x>0,p>0,q>0$ then

$$IH_1^{\beta}(X,t_1,t_2)=\frac{1}{\beta-1}\left(1-\frac{1}{p}\left[(1+pt_2)h_2^{\beta}(t_1,t_2)-(1+pt_1)h_1^{\beta}(t_1,t_2)\right]\right) \quad (22)$$

and

$$IH_2^{\beta}(X,t_1,t_2)=\frac{1}{1-\beta}\log\left(\frac{1}{p}\left[(1+pt_2)h_2^{\beta}(t_1,t_2)-(1+pt_1)h_1^{\beta}(t_1,t_2)\right]\right) \quad (23)$$

iv) power distribution with survival function $\overline{F}(x)=1-\left(\frac{x}{a}\right)^b;0<x<a,a>0,b>0$ then

$$IH_1^{\beta}(X,t_1,t_2)=\frac{1}{\beta-1}\left[1-\left(\frac{t_2}{\beta(b-1)+1}h_2^{\beta}(t_1,t_2)-\frac{t_1}{\beta(b-1)+1}h_1^{\beta}(t_1,t_2)\right)\right] \quad (24)$$

and

$$IH_2^{\beta}(X,t_1,t_2)=\frac{1}{1-\beta}\log\left[\frac{t_2}{\beta(b-1)+1}h_2^{\beta}(t_1,t_2)-\frac{t_1}{\beta(b-1)+1}h_1^{\beta}(t_1,t_2)\right] \quad (25)$$

3. Properties

In order to attain a decomposition of $H_1^{\beta}(X)$ and $H_2^{\beta}(X)$ similar to that given in Proposition 2.1 of Di Crescenzo and Longobardi [4] we have the following theorem.

Theorem 3.1. For a random lifetime X, $H_1^{\beta}(X)$ and $H_2^{\beta}(X)$ can be expressed as follows

$$\begin{aligned}H_1^{\beta}(X) &= \frac{1}{\beta-1}-\left[\frac{F^{\beta}(t_1)}{\beta-1}\left(1-(\beta-1)\overline{H}_1^{\beta}(X,t_1)\right)\right.\\&-\frac{(F(t_2)-F(t_1))^{\beta}}{\beta-1}\left(1-(1-\beta)IH_1^{\beta}(X,t_1,t_2)\right)\\&\left.-\frac{\overline{F}^{\beta}(t_2)}{\beta-1}\left((1-\beta)H_1^{\beta}(X,t_2)\right)\right],\end{aligned} \quad (26)$$

and

$$\begin{aligned}H_2^{\beta}(X) &= \frac{1}{1-\beta}\log\left[F^{\beta}(t_1)\exp\left((1-\beta)\overline{H}_2^{\beta}(X,t_1)\right)\right.\\&+(F(t_2)-F(t_1))^{\beta}\exp\left((1-\beta)IH_2^{\beta}(X,t_1,t_2)\right)\\&\left.+\overline{F}^{\beta}(t_2)\exp\left((1-\beta)H_2^{\beta}(X,t_2)\right)\right].\end{aligned} \quad (27)$$

Proof. Recalling (3), (6) and (12),

$$\begin{aligned} H_1^\beta(X) &= \frac{1}{\beta-1}\left(1-\int_0^\infty f^\beta(x)\,dx\right) \\ &= \frac{1}{\beta-1}\left(1-\left(\int_0^{t_1}(f(x))^\beta dx+\int_{t_1}^{t_2}(f(x))^\beta dx+\int_{t_2}^{\infty}(f(x))^\beta dx\right)\right) \\ &= \frac{1}{\beta-1}\Bigg(1-F^\beta(t_1)\int_0^{t_1}\left(\frac{f(x)}{F(t_1)}\right)^\beta dx \\ &\quad -(F(t_2)-F(t_1))^\beta\int_{t_1}^{t_2}\left(\frac{f(x)}{F(t_2)-F(t_1)}\right)^\beta dx \\ &\quad -\overline{F}^\beta(t_2)\int_t^\infty\left(\frac{f(x)}{\overline{F}(t_2)}\right)^\beta dx\Bigg), \end{aligned} \tag{28}$$

the other part is similar. ■

Similar to what given by Di Crescenzo and Longobardi [5], Theorem 4.1 can be interpreted in the following way. The uncertainty about the failure of a unit can be decomposed into four parts: first, the uncertainty about the failure time in $(0,t_1)$ such that the unit has failed before t_1; second, the uncertainty about the failure time in the interval (t_1,t_2) such that the unit has failed after t_1 but before t_2; third, the uncertainty about the failure time in $(t_2,+\infty)$ such that it has failed after t_2; and forth, the uncertainty of the random variable which determines if the unit has failed before t_1 or in between t_1 and t_2 or after t_2.

The following theorem is a characterization problem that explains the generalized interval entropy which determines the distribution function uniquely. One may get a similar kind of result in Belzunce et al. [3].

Remark 3.1. GFR functions determine distribution function uniquely. See Navarro and Ruiz [15].

Theorem 3.2. If X has an absolutely continuous distribution function $F(t)$ and if

(i) $IH_1^\beta(X,t_1,t_2)$ be increasing with respect to both coordinates t_1 and t_2, then $IH_1^\beta(X,t_1,t_2)$ uniquely determines $F(t)$.

(ii) $IH_2^\beta(X,t_1,t_2)$ be increasing with respect to both coordinates t_1 and t_2, then $IH_2^\beta(X,t_1,t_2)$ uniquely determines $F(t)$.

Proof. For proving item (i),

$$IH_1^\beta(X,t_1,t_2)=\frac{1}{\beta-1}\left(1-\int_{t_1}^{t_2}\left(\frac{f(x)}{F(t_2)-F(t_1)}\right)^\beta dx\right), \tag{29}$$

which implies that

$$\int_{t_1}^{t_2}(f(x))^\beta dx=(F(t_2)-F(t_1))^\beta\left(1-(\beta-1)\,IH_1^\beta(X,t_1,t_2)\right) \tag{30}$$

then by differentiating (30) with respect to both t_1 and t_2 and considering $h_j(t_1,t_2)=\frac{f(t_j)}{F(t_2)-F(t_1)}$,

$$h_1^\beta(t_1,t_2)=\beta h_1(t_1,t_2)-\beta(\beta-1)\,h_1(t_1,t_2)\,IH_1^\beta(X,t_1,t_2)+(\beta-1)\frac{\partial IH_1^\beta(X,t_1,t_2)}{\partial t_1} \tag{31}$$

and

$$h_2^\beta(t_1,t_2)=\beta h_2(t_1,t_2)-\beta(\beta-1)\,h_2(t_1,t_2)\,IH_1^\beta(X,t_1,t_2)-(\beta-1)\frac{\partial IH_1^\beta(X,t_1,t_2)}{\partial t_2}. \tag{32}$$

Hence, for fixed and positive t_1 and t_2, $h_1(t_1,t_2)$ and $h_2(t_1,t_2)$ are solutions of $g(x_{t_2})=0$ and $k(y_{t_1})=0$ where,

$$g(x_{t_2}) \stackrel{def}{=} \beta x_{t_2} - \beta(\beta-1)\, IH_1^{\beta}(X,t_1,t_2)\, x_{t_2} + (\beta-1)\frac{\partial IH_1^{\beta}(X,t_1,t_2)}{\partial t_1} - x_{t_2}^{\beta} \tag{33}$$

and

$$k(y_{t_1}) \stackrel{def}{=} \beta y_{t_1} - \beta(\beta-1)\, IH_1^{\beta}(X,t_1,t_2)\, y_{t_1} - (\beta-1)\frac{\partial IH_1^{\beta}(X,t_1,t_2)}{\partial t_2} - y_{t_1}^{\beta}. \tag{34}$$

Differentiating (33) and (34) with respect to x_{t_2} and y_{t_1}, give

$$\frac{\partial g(x_{t_2})}{\partial x_{t_2}} = \beta - \beta(\beta-1)\, IH_1^{\beta}(X,t_1,t_2) + (\beta-1) - \beta x_{t_2}^{\beta-1} \tag{35}$$

and

$$\frac{\partial k(y_{t_1})}{\partial y_{t_1}} = \beta - \beta(\beta-1)\, IH_1^{\beta}(X,t_1,t_2) + (\beta-1) - \beta y_{t_1}^{\beta-1}. \tag{36}$$

Now, $\frac{\partial g(x_{t_2})}{\partial x_{t_2}} = 0$ gives $x_1 = \left(1-(\beta-1)\, IH_1^{\beta}(X,t_1,t_2)\right)^{\frac{1}{\beta-1}}$ and $\frac{\partial k(y_{t_1})}{\partial y_{t_1}} = 0$ gives $y_1 = \left(1-(\beta-1)\, IH_1^{\beta}(X,t_1,t_2)\right)^{\frac{1}{\beta-1}}$.

Case I: If $\beta > 1$ then $g(0) = (\beta-1)\frac{\partial IH_1^{\beta}(X,t_1,t_2)}{\partial t_1} > 0$ and if $IH_1^{\beta}(X,t_1,t_2)$ is increasing in both coordinates t_1 and t_2, then $g(\infty)=\infty$. Further it can be seen that

$$\frac{\partial^2 g(x_{t_2})}{\partial x_{t_2}^2} = -\beta(\beta-1)\, x_{t_2}^{\beta-2} \le 0. \tag{37}$$

Therefore, $\frac{\partial g(x_{t_2})}{\partial x_{t_2}}$ is increasing in x_{t_2} and $g'(x_1)=0$, $g'(\infty)=-\infty$. Thus we see that

$$\begin{cases} g'(x_{t_2}) \ge 0; & 0 < x_{t_2} < x_1 \\ g'(x_{t_2}) < 0; & x_{t_2} > x_1. \end{cases} \tag{38}$$

In the same way, $k(0) > 0$ and if $IH_1^{\beta}(X,t_1,t_2)$ is increasing in both coordinates t_1 and t_2, then $k(\infty)=\infty$. Also $\frac{\partial^2 k(y_{t_1})}{\partial y_{t_1}^2} \le 0$ and $k'(y_1)=0$, $k'(\infty)=-\infty$. So

$$\begin{cases} k'(y_{t_1}) \ge 0; & 0 < y_{t_1} < y_1 \\ k'(y_{t_1}) < 0; & y_{t_1} > y_1. \end{cases} \tag{39}$$

Therefore, $g(x_{t_2})=0$ and $k(y_{t_1})=0$ have unique roots $h_1(t_1,t_2)$ and $h_2(t_1,t_2)$.

Case II: If $\beta < 1$ then $g(0) < 0$ and if $IH_1^{\beta}(X,t_1,t_2)$ is increasing in both coordinates t_1 and t_2, $g(\infty)=-\infty$. Further, it can be seen that $\frac{\partial^2 g(x_{t_2})}{\partial x_{t_2}^2} \ge 0$ and $g'(x_1)=0$, $g'(\infty)=\infty$. Therefore,

$$\begin{cases} g'(x_{t_2}) \le 0; & 0 < x_{t_2} < x_1 \\ g'(x_{t_2}) > 0; & x_{t_2} > x_1. \end{cases} \tag{40}$$

By the same argument, $k(0) < 0$ and if $IH_1^{\beta}(X,t_1,t_2)$ is increasing in both coordinates t_1 and t_2, then $k(\infty)=-\infty$. Also $\frac{\partial^2 k(y_{t_1})}{\partial y_{t_1}^2} \ge 0$ and $k'(y_1)=0$, $k'(\infty)=\infty$. Thus we have

$$\begin{cases} k'(y_{t_1}) \le 0; & 0 < y_{t_1} < y_1 \\ k'(y_{t_1}) > 0; & y_{t_1} > y_1. \end{cases} \tag{41}$$

Therefore, $g\left(x_{t_2}\right)=0$ and $k\left(y_{t_1}\right)=0$ have unique roots $h_1\left(t_1,t_2\right)$ and $h_2\left(t_1,t_2\right)$.

From two cases above, it can be concluded that if $IH_1^{\beta}\left(X,t_1,t_2\right)$ is increasing in both coordinates t_1 and t_2 and if $g\left(x_1\right)=0$ and $k\left(y_1\right)=0$, then $h_1\left(t_1,t_2\right)$ and $h_2\left(t_1,t_2\right)$ are the unique solutions of $g\left(x_{t_2}\right)=0$ and $k\left(y_{t_1}\right)=0$. So $IH_1^{\beta}\left(X,t_1,t_2\right)$ determines $h_j\left(t_1,t_2\right)$; $j=1,2$ uniquely. Again, due to Remark 3.1, $h_j\left(t_1,t_2\right)$; $j=1,2$ uniquely determine distribution function.

To prove (ii), from (15) we have

$$\exp\Big(\left(1-\beta\right)IH_2^{\beta}\left(X,t_1,t_2\right)\Big)=\int_{t_1}^{t_2}\left(\frac{f\left(x\right)}{F\left(t_2\right)-F\left(t_1\right)}\right)^{\beta}dx \tag{42}$$

differentiating both sides with respect to t_1 and t_2, we get

$$\left(1-\beta\right)\frac{\partial IH_2^{\beta}\left(X,t_1,t_2\right)}{\partial t_1}=\beta h_1\left(t_1,t_2\right)\exp\Big(\left(1-\beta\right)IH_2^{\beta}\left(X,t_1,t_2\right)\Big)-h_1^{\beta}\left(t_1,t_2\right) \tag{43}$$

$$\left(1-\beta\right)\frac{\partial IH_2^{\beta}\left(X,t_1,t_2\right)}{\partial t_2}=h_2^{\beta}\left(t_1,t_2\right)-\beta h_2\left(t_1,t_2\right)\exp\Big(\left(1-\beta\right)IH_2^{\beta}\left(X,t_1,t_2\right)\Big) \tag{44}$$

So for fixed t_1 and arbitrary t_2, $h_1\left(t_1,t_2\right)$ is a positive solution of the following equation

$$g\left(x_{t_2}\right)=x_{t_2}\frac{\beta\exp\Big(\left(1-\beta\right)IH_2^{\beta}\left(X,t_1,t_2\right)\Big)}{1-\beta}-\frac{\partial IH_2^{\beta}\left(X,t_1,t_2\right)}{\partial t_1}-\frac{x_{t_2}^{\beta}}{1-\beta}=0 \tag{45}$$

similarly, for fixed t_2 and arbitrary t_1, $h_2\left(t_1,t_2\right)$ is a positive solution of the following equation

$$k\left(y_{t_1}\right)=\frac{y_{t_1}^{\beta}}{1-\beta}-y_{t_1}\frac{\beta\exp\Big(\left(1-\beta\right)IH_2^{\beta}\left(X,t_1,t_2\right)\Big)}{1-\beta}-\frac{\partial IH_2^{\beta}\left(X,t_1,t_2\right)}{\partial t_2}=0. \tag{46}$$

Now, $\frac{\partial g\left(x_{t_2}\right)}{\partial x_{t_2}}=0$ gives $x_1=\left(\exp\Big(\left(1-\beta\right)IH_2^{\beta}\left(X,t_1,t_2\right)\Big)\right)^{\frac{1}{\beta-1}}$ and $\frac{\partial k\left(y_{t_1}\right)}{\partial y_{t_1}}=0$ gives $y_1=\left(\exp\Big(\left(1-\beta\right)IH_2^{\beta}\left(X,t_1,t_2\right)\Big)\right)^{\frac{1}{\beta-1}}$. Furthermore, considering second-order derivation of g and k with respect to x_{t_2} and y_{t_1} we have

$$\frac{\partial^2 g\left(x_{t_2}\right)}{\partial x_{t_2}^2}=\beta x_{t_2}^{\beta-2}, \tag{47}$$

and

$$\frac{\partial^2 k\left(y_{t_1}\right)}{\partial y_{t_1}^2}=-\beta y_{t_1}^{\beta-2}. \tag{48}$$

Again, $g\left(0\right)=-\frac{\partial IH_2^{\beta}\left(X,t_1,t_2\right)}{\partial t_1}$ and $k\left(0\right)=-\frac{\partial IH_2^{\beta}\left(X,t_1,t_2\right)}{\partial t_2}$.

Case I: $\left(\beta>1\right)$, $g\left(0\right)<0$ if $IH_2^{\beta}\left(X,t_1,t_2\right)$ is increasing in both coordinates t_1 and t_2 and $g\left(\infty\right)=\infty$. Similarly, one can say that $g\left(x_{t_2}\right)=0$ has a unique solution. Also, $k\left(0\right)<0$ and $k\left(\infty\right)=-\infty$ and $\frac{\partial^2 k\left(y_{t_1}\right)}{\partial y_{t_1}^2}<0$ i.e. $k\left(y_{t_1}\right)$ has a unique solution. Therefore, $g\left(x_{t_2}\right)=0$ and $k\left(y_{t_1}\right)=0$ have unique roots $h_1\left(t_1,t_2\right)$ and $h_2\left(t_1,t_2\right)$ respectively.

Case II: $(\beta<1)$, $g\left(0\right)<0$ and $k\left(0\right)<0$ if $IH_2^{\beta}\left(X,t_1,t_2\right)$ is increasing in both coordinates t_1 and t_2. In the same way one can conclude that $h_1\left(t_1,t_2\right)$ and $h_2\left(t_1,t_2\right)$ are unique solutions of $g\left(x_{t_2}\right)=0$ and $k\left(y_{t_1}\right)=0$ respectively.

From the above cases, it can be verified that if $IH_2^{\beta}(X,t_1,t_2)$ is increasing in both coordinates t_1 and t_2 and if $g(x_1)=0$ and $k(y_1)=0$, then $h_1(t_1,t_2)$ and $h_2(t_1,t_2)$ are the unique solutions of $g(x_{t_2})=0$ and $k(y_{t_1})=0$. So $IH_2^{\beta}(X,t_1,t_2)$ determines $h_j(t_1,t_2)$; $j=1,2$ uniquely. Now, by virtue of Remark 3.1, $h_j(t_1,t_2)$; $j=1,2$ determine distribution uniquely. ■

Remark 3.2. Since the generalized interval entropy determines the distribution function uniquely for each β, a natural question becomes apparent in this context is which β should be used in practice. The choice of β depends on the situation. For example, $IH_2^{\beta}(X,t_1,t_2)$ with $\beta=2$ could be used as a measure of economic diversity in the context, of income analysis. For more details see Abraham and Sankaran [1].

Theorem 3.3. The uniform distribution over (a,b), $a<b$ can be characterized by decreasing

i) First kind of generalized interval entropy $IH_1^{\beta}(X,t_1,t_2)=(1-\beta)^{-1}(1-(t_2-t_1))^{-(\beta-1)}$,

ii) Second kind of generalized interval entropy $IH_2^{\beta}(X,t_1,t_2)=\log(t_2-t_1)$.

Proof. For the first part, if $IH_1^{\beta}(X,t_1,t_2)$ is decreasing in both coordinates t_1 and t_2, then $g(x_{t_2})=0$ and $k(y_{t_1})=0$ have unique solutions so $g(x_1)=0$ and $k(y_1)=0$. The other part is similar. ■

Remark 3.3. If $IH_1^{\beta}(X,t_1,t_2)$ (respectively $IH_2^{\beta}(X,t_1,t_2)$) is decreasing in both coordinates t_1 and t_2 and $g(x_1)$ (respectively $k(y_1)$)$\neq 0$, then $g(x_{t_2})$ (respectively $k\left(y_{t_1}\right)$)$=0$ has two solutions for all positive t_1 and t_2. From these solutions, at least one should be GFR.

Example 3.1. If X has beta distribution with density function $f(t)=2x$, $0\leq x\leq 1$. Then for $\beta=2$, $IH_1^{\beta}(X,t_1,t_2)=1-\frac{4\left(t_2^3-t_1^3\right)}{3\left(t_2^2-t_1^2\right)^2}$ decreases for $t_1,t_2\in(0,1)$. Also by considering $h_1(t_1,t_2)$ as GFR function of X, we have

$$\frac{x_1}{h_1(t_1,t_2)}=\frac{2}{3}\left[\frac{t_2^2}{t_1(t_1+t_2)}+1\right]>1 \tag{49}$$

for $t_1,t_2\in(0,1)$. So, for every $t_1,t_2>0$, $g(x_{t_2})=0$ or $k\left(y_{t_1}\right)=0$ has two positive solutions as $h_1(t_1,t_2)$ and $h_1^*(t_1,t_2)$ such that $h_1(t_1,t_2)<x_1<h_1^*(t_1,t_2)$ and therefore $h_1^*(t_1,t_2)$ must be a GFR.

Again, for $\beta=2$, $IH_2^{\beta}(X,t_1,t_2)=-\log\frac{4\left(t_2^3-t_1^3\right)}{3\left(t_2^2-t_1^2\right)^2}$, which is decreasing for $t_1,t_2\in(0,1)$. Also,

$$\frac{x_1}{h_1(t_1,t_2)}=\frac{2}{3}\left[\frac{t_2^2}{t_1(t_1+t_2)}+1\right]>1 \tag{50}$$

for $t_1,t_2\in(0,1)$. So, in the same manner, both roots of $g(x_{t_2})=0$ or $k\left(y_{t_1}\right)=0$ are GFR. $IH_1^{\beta}(X,t_1,t_2)$ and $IH_2^{\beta}(X,t_1,t_2)$ are shown in Figure 1.

Theorem 3.4. The distribution of X is double truncated exponential if and only if $IH_1^{\beta}(X,t_1,t_2)(IH_2^{\beta}(X,t_1,t_2))=c$, where c is a constant.

Proof. As shown in (18), $IH_1^{\beta}(X,t_1,t_2)$ is constant. conversely, if $IH_1^{\beta}(X,t_1,t_2)=c$, (31) and (32) implies that

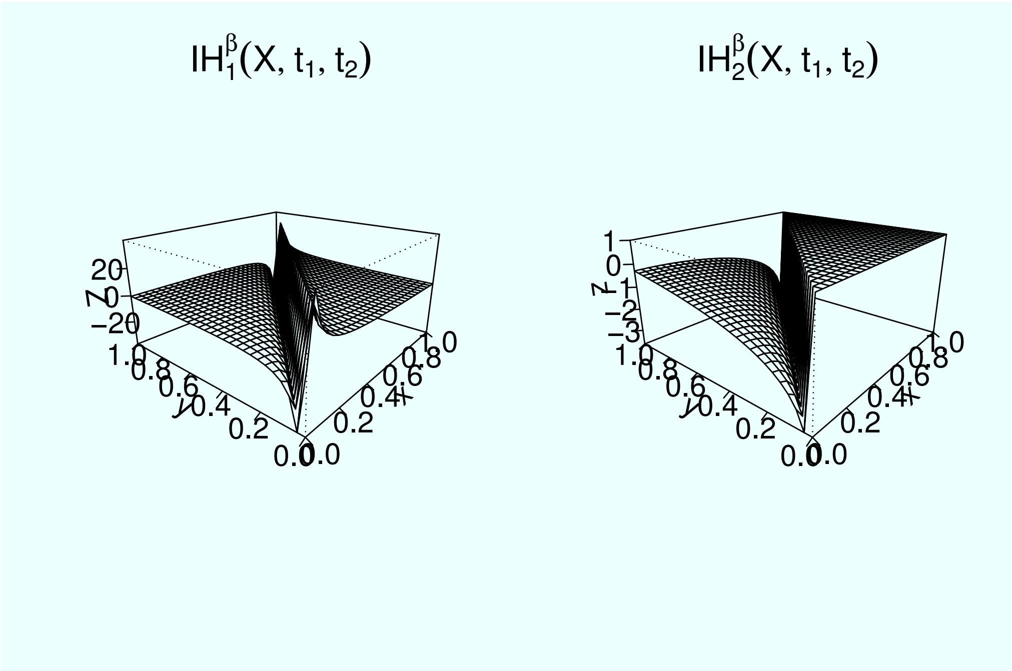
$$h_1^{\beta}(t_1,t_2)=h_1(t_1,t_2)\left(1-(\beta-1)\,IH_1^{\beta}(X,t_1,t_2)\right) \tag{51}$$

and

$$h_2^{\beta}(t_1,t_2)=h_2(t_1,t_2)\left(1-(\beta-1)\,IH_1^{\beta}(X,t_1,t_2)\right), \tag{52}$$

then $h_1^{\beta}(t_1,t_2)=h_2^{\beta}(t_1,t_2)=0$. Consequently, X is a double truncated exponential distribution. ■

Figure 1. The surface plot of the $IH_1^{\beta}(X, t_1, t_2)$ and $IH_2^{\beta}(X, t_1, t_2)$ in Example 3.1



Theorem 3.5. If X has an absolutely continuous distribution function $F(t)$, then a relationship of the form

$$IH_1^{\beta}(X, t_1, t_2) = \frac{1}{\beta - 1}\left(1 - \frac{1}{k}\left[(1 + ct_2)h_2^{\beta}(t_1, t_2) - (1 + ct_1)h_1^{\beta}(t_1, t_2)\right]\right) \tag{53}$$

and

$$IH_2^{\beta}(X, t_1, t_2) = \frac{1}{1 - \beta}\log\left(\frac{1}{k}\left[(1 + ct_2)h_2^{\beta}(t_1, t_2) - (1 + ct_1)h_1^{\beta}(t_1, t_2)\right]\right) \tag{54}$$

where k is constant holds for all $(t_1, t_2) \in D$ if and only if X follows exponential with $\bar{F}(x) = e^{-\theta x}$; $x > 0, \theta > 0$ for $c = 0$, Pareto distribution with $\bar{F}(x) = (1 + px)^{-q}; x > 0, p > 0, q > 0$, for $c > 0$ and finite range distribution with $\bar{F}(x) = (1 - ax)^{b}; 0 < x < \frac{1}{a}, a > 0, b > 0$ for $c < 0$.

Proof. Assume that the relation (53) holds. Then from the definitions of $h_i(t_1, t_2)$, and $IH_1^{\beta}(X, t_1, t_2)$, we can write (53) as

$$\int_{t_1}^{t_2} (f(x))^{\beta}\, dx = \frac{1}{k}\left[(1 + ct_2)f^{\beta}(t_2) - (1 + ct_1)f^{\beta}(t_1)\right] \tag{55}$$

differentiating with respect to t_i, $i = 1, 2$ and simplifying we get

$$\frac{f'(t_i)}{f(t_i)} = \frac{k - c}{\beta(1 + ct_i)}. \tag{56}$$

From (56) we get that X follows exponential, Pareto II and finite range distributions according as $c = 0$, $c > 0$, and $c < 0$. The converse part is obtained in example 2.1. Proof for $IH_2^{\beta}(X, t_1, t_2)$ is similar. ■

4. Some orders based on generalized interval entropy

In this section, we describe a few orders based on the generalized interval entropies and show their properties.

Proposition 4.1. Let X be an absolutely continuous random variable with density $f(x)$ and cumulative distribution function $F(x)$. Then

(i) increasing $h_1(t_1,t_2)$ in t_1 implies

$$IH_1^{\beta}(X,t_1,t_2) \leq \frac{1}{\beta-1}\left(1-h_1^{\beta}(t_1,t_2)\right) \tag{57}$$

and

$$IH_2^{\beta}(X,t_1,t_2) \geq \frac{1}{1-\beta}\log h_1^{\beta}(t_1,t_2) \tag{58}$$

(ii) decreasing $h_2(t_1,t_2)$ in t_2 implies

$$IH_1^{\beta}(X,t_1,t_2) \leq \frac{1}{\beta-1}\left(1-h_2^{\beta}(t_1,t_2)\right) \tag{59}$$

and

$$IH_2^{\beta}(X,t_1,t_2) \geq \frac{1}{1-\beta}\log h_2^{\beta}(t_1,t_2) \tag{60}$$

Proof. By recalling (14),

$$\begin{aligned} IH_1^{\beta}(X,t_1,t_2) &= \frac{1}{\beta-1}\left(1-\int_{t_1}^{t_2}\left(\frac{f(x)}{F(t_2)-F(t_1)}\right)^{\beta}dx\right) \\ &= \frac{1}{\beta-1}\left(1-\int_{t_1}^{t_2}\left(\frac{f(x)}{F(t_2)-F(x)}\frac{F(t_2)-F(x)}{F(t_2)-F(t_1)}\right)^{\beta}dx\right) \\ &= \frac{1}{\beta-1}\left(1-\int_{t_1}^{t_2}h_1^{\beta}(x,t_2)\left(\frac{F(t_2)-F(x)}{F(t_2)-F(t_1)}\right)^{\beta}dx\right). \end{aligned} \tag{61}$$

Because $\frac{F(t_2)-F(x)}{F(t_2)-F(t_1)} \geq 0$ and $t_1 < x$ implies that $h_1(x,t_2) \geq h_1(t_1,t_2)$. Then

$$\begin{aligned} IH_1^{\beta}(X,t_1,t_2) &\leq \frac{1}{\beta-1}\left(1-\int_{t_1}^{t_2}h_1^{\beta}(t_1,t_2)\left(\frac{F(t_2)-F(x)}{F(t_2)-F(t_1)}\right)^{\beta}dx\right) \\ &= \frac{1}{\beta-1}\left[1-h_1^{\beta}(t_1,t_2)\int_{t_1}^{t_2}\left(\frac{F(t_2)-F(x)}{F(t_2)-F(t_1)}\right)^{\beta}dx\right] \\ &\leq \frac{1}{\beta-1}\left(1-h_1^{\beta}(t_1,t_2)\right). \end{aligned} \tag{62}$$

Also, recalling (15) and using same argument as above, we have

$$\begin{aligned} IH_2^{\beta}(X,t_1,t_2) &= \frac{1}{1-\beta}\log\left(\int_{t_1}^{t_2}\left(\frac{f(x)}{F(t_2)-F(t_1)}\right)^{\beta}dx\right) \\ &= \frac{1}{1-\beta}\log\left(\int_{t_1}^{t_2}h_1^{\beta}(x,t_2)\left(\frac{F(t_2)-F(x)}{F(t_2)-F(t_1)}\right)^{\beta}dx\right). \end{aligned} \tag{63}$$

In the same manner $\frac{F(t_2)-F(x)}{F(t_2)-F(t_1)} \geq 0$ and $t_1 < x$ implies that $h_1(x, t_2) \geq h_1(t_1, t_2)$. So

$$\begin{aligned} IH_2^{\beta}(X, t_1, t_2) &\geq \frac{1}{1-\beta} \log \left(\int_{t_1}^{t_2} h_1^{\beta}(t_1, t_2) \left(\frac{F(t_2) - F(x)}{F(t_2) - F(t_1)} \right)^{\beta} dx \right) \\ &= \frac{1}{1-\beta} \left[\log h_1^{\beta}(t_1, t_2) + \log \left(\int_{t_1}^{t_2} \left(\frac{F(t_2) - F(x)}{F(t_2) - F(t_1)} \right)^{\beta} dx \right) \right] \\ &\geq \frac{1}{1-\beta} \log h_1^{\beta}(t_1, t_2). \end{aligned} \tag{64}$$

The proof of the second part is similar. ■

In the following example we consider the case of identical GFR function.

Example 4.1. For $\beta = 2$, if X has Uniform distribution on the interval (a, b) with $f(x) = \frac{1}{b-a}$ and $h_1(t_1, t_2) = h_2(t_1, t_2) = \frac{1}{t_2 - t_1}$, then $IH_1^{\beta}(X, t_1, t_2) = 1 - \frac{1}{(t_2 - t_1)}$ and $IH_2^{\beta}(X, t_1, t_2) = -\log(t_2 - t_1)$. By recalling Proposition 4.1, we see that relationship is valid.

It must be mentioned that in Proposition 4.1, first (second) kind of interval entropy depends on only one of the GFR functions. Example 2.1 showed that $IH_1^{\beta}(X, t_1, t_2)$ ($IH_2^{\beta}(X, t_1, t_2)$) depends on both GFR function.

In the sequel, we give a definition in agreement with Khorashadizadeh et al. [10].

Definition 4.1. The random variable X is said to have

i) decreasing first kind interval entropy or (DFIE) property if and only if for fixed t_2, $IH_1^{\beta}(X, t_1, t_2)$ is decreasing with respect to t_1.

ii) decreasing second kind interval entropy or (DSIE) property if and only if for fixed t_2, $IH_2^{\beta}(X, t_1, t_2)$ is decreasing with respect to t_1.

This implies that $IH_i^{\beta}(X, t_1, t_2)$; $i = 1, 2$, has DFIE(DSIE) if $\frac{\partial IH_i^{\beta}(X, t_1, t_2)}{\partial t_1} \leq 0$.

Theorem 4.1. If X is a nonnegative random variable then $IH_i^{\beta}(X, t_1, t_2)$; $i = 1, 2$ cannot be increasing function with respect to t_1 for any fixed t_2.

Proof. First note that, using Hopital's rule we have

$$\begin{aligned} \lim_{t_1 \to t_2} IH_1^{\beta}(X, t_1, t_2) &= \lim_{t_1 \to t_2} \frac{1}{\beta - 1} \left(1 - \int_{t_1}^{t_2} \left(\frac{f(x)}{F(t_2) - F(t_1)} \right)^{\beta} dx \right) \\ &= \lim_{t_1 \to t_2} \frac{1}{\beta - 1} - \lim_{t_1 \to t_2} \int_{t_1}^{t_2} \left(\frac{f(x)}{F(t_2) - F(t_1)} \right)^{\beta} dx \\ &= \frac{1}{\beta - 1} - \lim_{t_1 \to t_2} \frac{1}{\beta(\beta - 1)} \left(\frac{f(t_1)}{F(t_2) - F(t_1)} \right)^{\beta - 1} \\ &= -\infty, \end{aligned} \tag{65}$$

Now, on the contrary suppose that $IH_1^{\beta}(X, t_1, t_2)$ is increasing in t_1, then for all $t_1 \leq t_2$, $IH_1^{\beta}(X, t_1, t_2) \leq IH_1^{\beta}(X, t_2, t_2) = -\infty$ which contradicts the fact that $IH_1^{\beta}(X, t_1, t_2) \in R$ for all $(t_1, t_2) \in D$.

In similar manner, we can conclude that $IH_2^{\beta}(X, t_1, t_2)$ is non increasing.

Theorem 4.2. Let X be a nonnegative random variable with probability density function $f(x)$ and cumulative function $F(x)$ then

$$\text{i) } IH_1^{\beta}(X,t_1,t_2) \leq \frac{1}{\beta-1}\left(1-\frac{1}{\beta}\left(\frac{1+\frac{\partial\mu(t_1,t_2)}{\partial t_1}}{\mu(t_1,t_2)}\right)^{\beta-1}\right) \tag{66}$$

and

$$\text{ii) } IH_2^{\beta}(X,t_1,t_2) \leq \frac{1}{\beta-1}\log\frac{1}{\beta}\left(\frac{1+\frac{\partial\mu(t_1,t_2)}{\partial t_1}}{\mu(t_1,t_2)}\right)^{\beta-1} \tag{67}$$

where

$$\mu(t_1,t_2) = E(X-t|t_1<X<t_2) = \frac{1}{F(t_2)-F(t_1)}\int_{t_1}^{t_2}(z-t_1)\,dF(z) \tag{68}$$

is the doubly truncated mean residual life function

Proof. For proving item (i) note that,

$$\frac{\partial IH_1^{\beta}(X,t_1,t_2)}{\partial t_1} = \frac{1}{\beta-1}\left(h_1^{\beta}(t_1,t_2)-\beta h_1(t_1,t_2)\int_{t_1}^{t_2}\left(\frac{f(x)}{F(t_2)-F(t_1)}\right)^{\beta}dx\right) \tag{69}$$

if substitute $h_1(t_1,t_2) = \frac{1+\frac{\partial\mu(t_1,t_2)}{\partial t_1}}{\mu(t_1,t_2)}$ then

$$\begin{aligned}\frac{\partial IH_1^{\beta}(x,t_1,t_2)}{\partial t_1} &= \frac{1}{\beta-1}\left[\left(\frac{1+\frac{\partial\mu(t_1,t_2)}{\partial t_1}}{\mu(t_1,t_2)}\right)^{\beta}+\beta\left(\frac{1+\frac{\partial\mu(t_1,t_2)}{\partial t_1}}{\mu(t_1,t_2)}\right)IH_1^{\beta}(X,t_1,t_2)\right.\\ &\qquad \left.-\frac{\beta}{\beta-1}\left(\frac{1+\frac{\partial\mu(t_1,t_2)}{\partial t_1}}{\mu(t_1,t_2)}\right)\right]\end{aligned} \tag{70}$$

which satisfied the first result.

For proving (ii),

$$\frac{\partial IH_2^{\beta}(X,t_1,t_2)}{\partial t_1} = \beta h_1(t_1,t_2)\int_{t_1}^{t_2}\left(\frac{f(x)}{F(t_2)-F(t_1)}\right)^{\beta}dx - h_1^{\beta}(t_1,t_2) \leq 0. \tag{71}$$

This satisfied the second result. ■

Proposition 4.2. In Theorem 4.2, as $t_2 \to \infty$, we have that $IH_i^{\beta}(X,t)$; $i = 1,2$ is increasing (decreasing) with respect to t, if and only if the following inequalities hold for all $t > 0$.

$$H_1^{\beta}(X,t) \leq (\geq)\,\frac{1}{\beta-1}-\frac{h^{\beta-1}(t)}{\beta(\beta-1)}, \tag{72}$$

and

$$H_2^{\beta}(X,t) \leq (\geq)\,\frac{h^{\beta-1}(t)}{\beta(\beta-1)}, \tag{73}$$

where $h(t) = \frac{1+\acute{\mu}(t)}{\mu(t)}$.

Proof. Using (6),

$$\frac{\partial H_1^{\beta}(X,t)}{\partial t} = \frac{1}{\beta-1}\left(h^{\beta}(t)-\beta h(t)\int_t^{\infty}\left(\frac{f(x)}{\overline{F}(t)}\right)^{\beta}dx\right). \tag{74}$$

If $H_1^{\beta}(X,t)$ is increasing in $t,$ then $H_1^{\beta}(X,t) \geq 0$ i.e $H_1^{\beta}(X,t) \geq \frac{1}{\beta-1}\left(1-\frac{h^{\beta-1}(t)}{\beta}\right)$. If $H_1^{\beta}(X,t)$ decreased in $t,$ then $H_1^{\beta}(X,t) \leq 0$ i.e $H_1^{\beta}(X,t) \leq \frac{1}{\beta-1}\left(1-\frac{h^{\beta-1}(t)}{\beta}\right)$. Therefore the first result obtained. The second part is similar.

5. Conclusion

In literature of information measures, generalized interval entropy is a famous concept which always give a nonnegative uncertainty measure. But in many survival studies for modeling statistical data, information about lifetime between two points is available. Considering, the concept of doubly truncated (interval) entropy has been introduced. In this paper, several results on the first and second kind of generalized interval entropies have been discussed. Also, it has been shown that generalized interval entropies determine the distribution of random variables uniquely. Some orders based on given uncertainty measures have been given.

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