Read-Bajraktarević functional in higher order Orlicz-Sobolev space

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Abstract

We justify a step in the proof of a theorem recently published, which states the local fractal functions of an Orlicz-Sobolev class of order $m \geq 1$ appear naturally as the fixed points of the restriction of the Read-Bajraktarević operators. The later was well-known (since the works of Massopust *et al.*) in the context of Lebesgue and Sobolev spaces. We also relaxed a condition (inequality) that ensures contractivity of the Read-Bajraktarević functional, and prove that the number q_Φ that appears in the proof is not required to be greater than 1.

Young Function $\Phi : [0,\infty) \to [0,\infty)$

1. Φ is a convex function $\iff \{(t,y) \in [0,\infty) \times [0,\infty) : y \geq \Phi(t)\}$ is a convex set
2. $\Phi(0) = 0, \lim\limits_{t\to\infty} \Phi(t) = \infty, \lim\limits_{t\to 0} \frac{\Phi(t)}{t} = 0, \lim\limits_{t\to\infty} \frac{\Phi(t)}{t} = \infty$
3. Φ is strictly increasing, continuous, unbounded with the integral representation

$$\Phi(t) = \int_0^t \phi(s)ds,$$

where $\phi : [0,\infty) \to [0,\infty)$ is a non-decreasing odd homeomorphism
4. Φ satisfies a Δ_2 condition, denoted as $\Phi \in \Delta_2$ i.e., $\forall r > 1, \exists \gamma(r) > 0; \Phi(rt) \leq \gamma(r)\Phi(t)$, for $t \geq T > 0$.
5. Assume there exists positive integers p_Φ and q_Φ such that

$$p_\Phi \leq \frac{t\phi(t)}{\Phi(t)} \leq q_\Phi, t \neq 0$$

, so we can obtain

$$\min\{\rho^{p_\Phi}, \rho^{q_\Phi}\}\Phi(t) \leq \Phi(\rho t) \leq \max\{\rho^{p_\Phi}, \rho^{q_\Phi}\}\Phi(t),$$

for any non negative ρ and positive t.

Orlicz-Sobolev space and Read-Bajraktarević functional

Let $X \subset \mathbb{R}^N$ be the nonempty connected and bounded set. The Orlicz class $\mathcal{L}_\Phi(X)$ is the set of equivalence classes of real valued measurable functions f such that $\Phi(|f|) \in L^1(X)$. The linear span $L_\Phi(X)$ of the Orlicz class is the complete normed vector space with the Lexumberg norm

$$||f||_\Phi = \inf\left\{\tau > 0 : \int_X \Phi\left(\frac{|f|}{\tau}\right) dx \leq 1\right\}$$

The Orlicz-Sobolve space of order m is defined as

$$W^m L_\Phi(X) = \{f \in L_\Phi(X) : D_x^{\boldsymbol{\beta}} f \in L_\Phi(X), |\boldsymbol{\beta}| \leq m, \boldsymbol{\beta} = (\beta_1, \dots, \beta_N) \in \mathbb{N}^N\}, D_x^{\boldsymbol{\beta}} f = \frac{\partial^{|\boldsymbol{\beta}|} f}{\partial x_1^{\beta_1} \cdots \partial x_N^{\beta_N}}.$$

This vector space is Banach with the norm

$$||f||_{m,\Phi} = \sum_{|\boldsymbol{\beta}| \leq m} ||D_x^{\boldsymbol{\beta}} f||_\Phi.$$

Let $\{X_i\}_{i=1}^n$ be a family of nonempty and connected subset of X, and $\{\alpha_i : X_i \to X\}$ be a collection of diffeomorphisms such that

$$\alpha_i(X_i) \bigcap \alpha_j(X_j) = \emptyset, i \neq j, \text{and } X = \bigcup_{i=1}^n \alpha_i(X_i).$$

We also assume one uniformly bounded condition for those mappings

$$b_i(m) := \max\{1, \max_{j\in[1,N], |\boldsymbol{\theta}| \leq m} \sup_{x \in X_i} |D_x^{\boldsymbol{\theta}} (\alpha_i^{-1})_j(x)|\} < \infty$$

up to the order m. Let $\{\lambda_i : X_i \to \mathbb{R}\} \subset W^m L_\Phi(X)$ and $\{R_i\}_{i=1}^n \subset \mathbb{R}$, the Read-Bajraktarević functional $\mathbf{T} : W^m L_\Phi(X) \to \mathbb{R}^X$ is the map

$$\mathbf{T}(u(x)) = \sum_{i=1}^n (\lambda_i \circ \alpha_i^{-1})(x) \mathbf{1}_{\alpha_i(X_i)}(x) + \sum_{i=1}^n R_i \cdot (u|_{X_i} \circ \alpha_i^{-1})(x) \mathbf{1}_{\alpha_i(X_i)}(x).$$

We also introduce the positive constants

$$a_i = \sup_{x \in X_i} |\det J_x \alpha_i|, M = \sum_{i=1}^n a_i r_i^{q_\Phi} (b_i(m))^{m q_\Phi},$$

where $r_i = \max\{1, |R_i|\}, i \in [1,n]$.

Observation of The Theorem

From a recently published paper, we can obtain this result: assume $1 \leq |\boldsymbol{\sigma}| \leq |\boldsymbol{\beta}| \leq m$, if $q_\Phi > 1$, then the Read-Bajraktarević functional is a contraction on $W^m L_\Phi(X)$, provided $M(N_m S_m)^{q_\Phi} < 1$. In the proof of this result, we obtain this estimate

$$\int_X \Phi\left(\frac{|D_x^{\boldsymbol{\beta}} \mathbf{T} u(x) - D_x^{\boldsymbol{\beta}} \mathbf{T} v(x)|}{\tau}\right) dx \leq M N_m^{q_\Phi - 1} S_m^{q_\Phi} \sum_{|\boldsymbol{\sigma}|=1}^m \int_X \Phi\left(\frac{|D_x^{\boldsymbol{\sigma}} u(x) - D_x^{\boldsymbol{\sigma}} v(x)|}{\tau}\right) dx,$$

$\forall u, v \in W^m L_\Phi(X)$ and N_m is the number of the multi-indices of $1 \leq |\boldsymbol{\sigma}| \leq m$. The convexity of the Young function gives us $\Phi(at) \leq a\Phi(t), \forall a \in [0,1], t \geq 0$, so

$$\sum_{|\boldsymbol{\sigma}|=1}^m \int_X \Phi\left(\frac{|D_x^{\boldsymbol{\sigma}} u(x) - D_x^{\boldsymbol{\sigma}} v(x)|}{\sum_{|\boldsymbol{\beta}|=1}^m ||D_x^{\boldsymbol{\beta}} u(x) - D_x^{\boldsymbol{\beta}} v(x)||_\Phi}\right) dx \leq \sum_{|\boldsymbol{\sigma}|=1}^m \frac{|||D_x^{\boldsymbol{\sigma}} u(x) - D_x^{\boldsymbol{\sigma}} v(x)||_\Phi}{\sum_{|\boldsymbol{\beta}|=1}^m ||D_x^{\boldsymbol{\beta}} u(x) - D_x^{\boldsymbol{\beta}} v(x)||_\Phi} \int_X \Phi\left(\frac{|D_x^{\boldsymbol{\sigma}} u(x) - D_x^{\boldsymbol{\sigma}} v(x)|}{|||D_x^{\boldsymbol{\sigma}} u(x) - D_x^{\boldsymbol{\sigma}} v(x)||_\Phi}\right) < 1,$$

so

$$||D_x^{\boldsymbol{\beta}} \mathbf{T} u(x) - D_x^{\boldsymbol{\beta}} \mathbf{T} v(x)||_\Phi \leq M N_m^{q_\Phi - 1} S_m^{q_\Phi} \sum_{|\boldsymbol{\beta}|=1}^m ||D_x^{\boldsymbol{\beta}} u(x) - D_x^{\boldsymbol{\beta}} v(x)||_\Phi,$$

$$||\mathbf{T} u(x) - \mathbf{T} v(x)||_{m,\Phi} \leq M N_m^{q_\Phi} S_m^{q_\Phi} ||u - v||_{m,\Phi},\ M(N_m S_m)^{q_\Phi} < 1$$

An Improved (Less Restrictive) Estimate

Let $g_i = u|_{X_i} - v|_{X_i}, y = \alpha_i^{-1}(x)$, we have already known $D_x^{\beta}(g_i \circ \alpha_i^{-1})(x) = \sum_{|\boldsymbol{\sigma}|=\mathbf{1}}^{|\boldsymbol{\beta}|} D_y^{\boldsymbol{\sigma}} g_i(y) \sum_{k=1}^{|\boldsymbol{\beta}|} \sum_{E_k(\boldsymbol{\beta},\boldsymbol{\sigma})} \boldsymbol{\beta}! \Pi_{j=1}^k \frac{(\mathbf{D}_x^{\boldsymbol{\theta}_j}(\alpha_i^{-1}(x))^{\boldsymbol{\mu}_j}}{\boldsymbol{\mu}_j!(\boldsymbol{\theta}_j!)^{|\boldsymbol{\mu}_j|}}$, and then

$$|D_x^{\beta}(g_i \circ \alpha_i^{-1})(x)| \leq \mathbf{S}_{\boldsymbol{\beta}}^{\boldsymbol{\sigma}} b_i(m)^{|\boldsymbol{\sigma}|} \sum_{|\boldsymbol{\sigma}|=\mathbf{1}}^{|\boldsymbol{\beta}|} |D_y^{\boldsymbol{\sigma}} g_i(y)| \leq b_i(m)^m S_m \sum_{|\boldsymbol{\sigma}|=\mathbf{1}}^{m} |D_y^{\boldsymbol{\sigma}} g_i(y)|,$$

where $S_m = \max_{0 \leq |\boldsymbol{\sigma}| \leq |\beta| \leq m} \mathbf{S}_{\boldsymbol{\beta}}^{\boldsymbol{\sigma}}$.

$$\int_X \Phi\left(\frac{|D_x^{\boldsymbol{\beta}} \mathbf{T} u(x) - D_x^{\boldsymbol{\beta}} \mathbf{T} v(x)|}{\tau}\right) dx \leq \sum_{i=1}^n \int_{\alpha_i(X_i)} \Phi\left(\frac{r_i |D_x^{\boldsymbol{\beta}} g_i \circ \alpha_i^{-1}(x)|}{\tau}\right) dx \leq \sum_{i=1}^n \int_{\alpha_i(X_i)} \Phi\left(\frac{r_i S_m b_i(m)^m \sum_{|\boldsymbol{\sigma}|=\mathbf{1}}^m |D_y^{\boldsymbol{\sigma}} g_i(y)|}{\tau}\right) dx,$$

and this is bounded by

$$\left(\sum_{i=1}^n a_i (r_i b_i(m)^m)^{q_\Phi}\right) S_m^{q_\Phi} \int_X \Phi\left(\frac{\sum_{|\boldsymbol{\sigma}|=1}^m |D_x^{\boldsymbol{\sigma}} u(x) - D_x^{\boldsymbol{\beta}} v(x)|}{\tau}\right) dx = M S_m^{q_\Phi} \int_X \Phi\left(\frac{\sum_{|\boldsymbol{\sigma}|=1}^m |D_x^{\boldsymbol{\sigma}} u(x) - D_x^{\boldsymbol{\beta}} v(x)|}{\tau}\right) dx,$$

so we can obtain

$$||\mathbf{T} u - \mathbf{T} v||_{m,\Phi} \leq M N_m S_m^{q_\Phi} ||u - v||_{m,\Phi}, M N_m S_m^{q_\Phi} < 1$$