Random Fields in Complex Systems Modelling: Information Geometry and Fisher Curves

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Overview

Introduction

Random fields are useful mathematical structures in the study and characterization of non-deterministic complex systems

The mais goal is to understand how relationships between pieces of information give rise to collective behaviors among different scale levels of a system

In complex systems, the interaction between the components is highly non-linear and/or non-deterministic

Introduction

Random fields are particularly interesting mathematical structures due to the following assumptions:

- it is possible to replace the usual statistical independence assumption by a more realistic conditional independence hypothesis
- we can restrict the size of the maximum clique to be two, that is, we can assume only binary relationships
- considering that the coupling parameter is invariant and isotropic, all the information regarding the spatial dependence structure of the random field is conveyed by a single scalar parameter - the inverse temperature β

The q-state Potts model

It is a model used to study collective effects based on consequences of local interactions in the case of a finite set of behaviors (q)

Definition

A pairwise q-state Potts random field regarding a local neighborhood system η_i defined on a lattice $S = \{s_1, s_2, \ldots, s_n\}$ is completely characterized by a set of n local conditional density functions $p(x_i|\eta_i, \beta)$, given by:

$$p(x_i = m|\eta_i, \beta) = \frac{exp\{\beta U_i(m)\}}{\sum_{l=1}^{q} exp\{\beta U_i(l)\}} \quad (1)$$

where $U_i(m)$ denotes the number of times the behavior m is found in the neighborhood of an element and β is the inverse temperature, a global parameter that controls the spatial dependence structure of the system.

Isotropic pairwise Gaussian random fields

The main advantage of this model is the mathematical tractability. It is a model suitable for modelling complex systems in which each cell may assume an infinite number of states at a given time

Definition

An isotropic pairwise Gaussian Markov random field regarding a local neighborhood system η_i defined on a lattice $S = \{s_1, s_2, \ldots, s_n\}$ is completely characterized by a set of n local conditional density functions $p(x_i|\eta_i, \vec{\theta})$, given by:

$$p\left(x_i|\eta_i, \vec{\theta}\right) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left\{-\frac{1}{2\sigma^2}\left[x_i - \mu - \beta\sum_{j\in\eta_i}(x_j - \mu)\right]^2\right\} \quad (2)$$

with $\vec{\theta} = (\mu, \sigma^2, \beta)$ the parameters vector, where μ and σ^2 are respectively the expected value (mean) and the variance of the random variables in the field, and β is the inverse temperature

Parameter Estimation in Random Fields

In order to compute statistical measures from a random field (entropy, Fisher information) it is necessary to estimate the model parameters.

However, maximum likelihood estimation can be intractable for the inverse temperature parameter estimation, due to the existence of the partition function in the joint Gibbs distribution.

An alternative, is to perform maximum pseudo-likelihood estimation, which is based on the conditional independence principle.

Maximum Pseudo-Likelihood Estimation

Definition

Let an isotropic pairwise Markov random field model be defined on a rectangular lattice $S = \{s_1, s_2, \ldots, s_n\}$ *with a neighborhood system* η_i. *Assuming that* $\mathbf{X^{(t)}} = \{x_1^{(t)}, x_2^{(t)}, \ldots, x_n^{(t)}\}$ *denotes the set corresponding to the observations at a time t (a snapshot of the random field), the pseudo-likelihood function of the model is defined by:*

$$L\left(\beta; \mathbf{X}^{(t)}\right) = \prod_{i=1}^{n} p(x_i|\eta_i, \beta) \tag{3}$$

The pseudo-likelihood function is the product of the local conditional density functions throughout the field. Note that the pseudo-likelihood function is a function of the model parameters.

Estimating the Inverse Temperature in the GMRF Model

The maximum pseudo-likelihood function is given by:

$$log\ L\left(\vec{\theta}; \mathbf{X}^{(t)}\right) = -\frac{n}{2} log\left(2\pi\sigma^2\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left[x_i - \mu - \beta\sum_{j\in\eta_i}(x_j - \mu)\right]^2 \quad (4)$$

Maximizing it in β and after some algebra we have:

$$\hat{\beta}_{MPL} = \frac{\sum_{j\in\eta_i}\hat{\sigma}_{ij}}{\sum_{j\in\eta_i}\sum_{k\in\eta_i}\hat{\sigma}_{jk}} \quad (5)$$

where $\hat{\sigma}_{ij}$ denotes the sample covariance between the central variable x_i and $x_j \in \eta_i$. Similarly, $\hat{\sigma}_{jk}$ denotes the sample covariance between two variables belonging to the neighborhood system η_i (the definition of η_i does not include the the location s_i).

Estimating the Inverse Temperature in the Potts Model

The pseudo-likelihood equation for the Potts model is given by:

$$L(\beta) = \prod_{i \in S} p(x_i = m_i \mid \eta_i) = \prod_{i \in S} \frac{\exp\{\beta U_i(m_i)\}}{\sum_{\ell=1}^{q} \exp\{\beta U_i(\ell)\}} \tag{6}$$

Taking the logarithms, differentiating on the parameter and setting the result to zero, leads to the following expression, which is the basis for the derivation of the proposed pseudo-likelihood equation:

$$\frac{\partial}{\partial \beta} log L(\beta) = \sum_{i \in S} U_i(m_i) - \sum_{i \in S} \left[\frac{\sum_{\ell=1}^{q} U_i(\ell) \exp\{\beta U_i(\ell)\}}{\sum_{\ell=1}^{q} \exp\{\beta U_i(\ell)\}} \right] = 0 \tag{7}$$

Estimating the Inverse Temperature in the Potts Model

The derivation of the q-state Potts model pseudo-likelihood equation consists in expanding the second term of previous equation in all possible spatial configuration patterns that provide different contributions

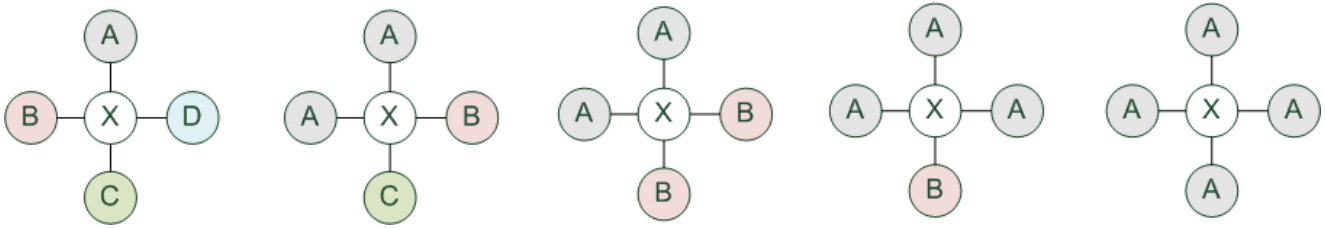

Figure : Contextual configuration patterns for Potts MRF model in first order neighborhood systems

In the Potts model location information is irrelevant since it is an isotropic model:

$$\vec{v}_0 = [1,1,1,1] \qquad \vec{v}_1 = [2,1,1,0] \qquad \vec{v}_2 = [2,2,0,0] \tag{8}$$

$$\vec{v}_3 = [3,1,0,0] \qquad \vec{v}_4 = [4,0,0,0] \tag{9}$$

Estimating the Inverse Temperature in the Potts Model

Note that the set of all possible configuration patterns given a neighborhood system of size K is composed by the set of vectors that define all the partitions of the integer K. It is known that this structure defines a poset, represented by the following Hasse diagram (for $K = 8$):

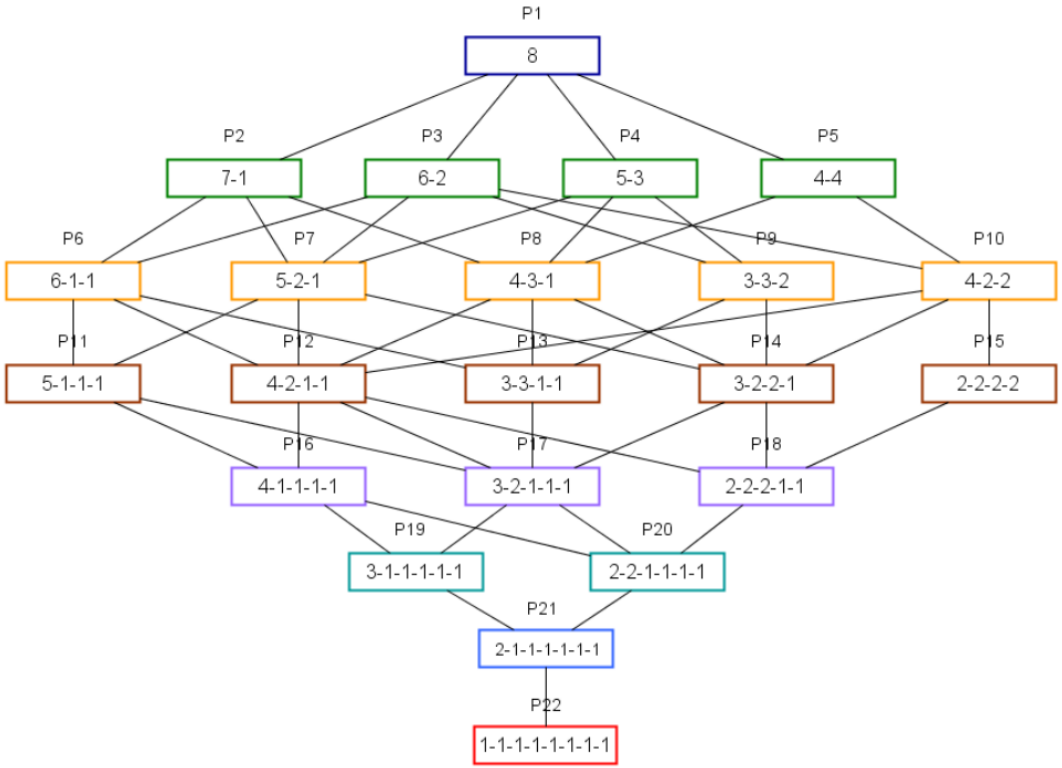

Figure : Note the highly non-linear topology, with several irregularities and asymmetries.

Maximum Pseudo-Likelihood Equation (Potts model)

$$\sum_{s\in\S} U_s\left(m_s\right) - \frac{8e^{8\hat{\beta}}}{e^{8\hat{\beta}}+q-1}K_1 - \frac{7e^{7\hat{\beta}}+e^{\hat{\beta}}}{e^{7\hat{\beta}}+e^{\hat{\beta}}+q-2}K_2 - \frac{6e^{6\hat{\beta}}+2e^{2\hat{\beta}}}{e^{6\hat{\beta}}+e^{2\hat{\beta}}+q-2}K_3$$
$$- \frac{6e^{6\hat{\beta}}+2e^{\hat{\beta}}}{e^{6\hat{\beta}}+2e^{\hat{\beta}}+q-3}K_4 - \frac{5e^{5\hat{\beta}}+3e^{3\hat{\beta}}}{e^{5\hat{\beta}}+e^{3\hat{\beta}}+q-2}K_5 - \frac{5e^{5\hat{\beta}}+2e^{2\hat{\beta}}+e^{\hat{\beta}}}{e^{5\hat{\beta}}+e^{2\hat{\beta}}+e^{\hat{\beta}}+q-3}K_6$$
$$- \frac{5e^{5\hat{\beta}}+3e^{\hat{\beta}}}{e^{5\hat{\beta}}+3e^{\hat{\beta}}+q-4}K_7 - \frac{8e^{4\hat{\beta}}}{2e^{4\hat{\beta}}+q-2}K_8 - \frac{4e^{4\hat{\beta}}+3e^{3\hat{\beta}}+e^{\hat{\beta}}}{e^{4\hat{\beta}}+e^{3\hat{\beta}}+e^{\hat{\beta}}+q-3}K_9$$
$$- \frac{4e^{4\hat{\beta}}+4e^{2\hat{\beta}}}{e^{4\hat{\beta}}+2e^{2\hat{\beta}}+q-3}K_{10} - \frac{4e^{4\hat{\beta}}+2e^{2\hat{\beta}}+2e^{\hat{\beta}}}{e^{4\hat{\beta}}+e^{2\hat{\beta}}+2e^{\hat{\beta}}+q-4}K_{11}$$
$$- \frac{4e^{4\hat{\beta}}+4e^{\hat{\beta}}}{e^{4\hat{\beta}}+4e^{\hat{\beta}}+q-5}K_{12} - \frac{6e^{3\hat{\beta}}+2e^{2\hat{\beta}}}{2e^{3\hat{\beta}}+e^{2\hat{\beta}}+q-3}K_{13} - \frac{6e^{3\hat{\beta}}+2e^{\hat{\beta}}}{2e^{3\hat{\beta}}+2e^{\hat{\beta}}+q-4}K_{14}$$
$$- \frac{3e^{3\hat{\beta}}+4e^{2\hat{\beta}}+e^{\hat{\beta}}}{e^{3\hat{\beta}}+2e^{2\hat{\beta}}+e^{\hat{\beta}}+q-4}K_{15} - \frac{3e^{3\hat{\beta}}+2e^{2\hat{\beta}}+3e^{\hat{\beta}}}{e^{3\hat{\beta}}+e^{2\hat{\beta}}+3e^{\hat{\beta}}+q-5}K_{16}$$
$$- \frac{3e^{3\hat{\beta}}+5e^{\hat{\beta}}}{e^{3\hat{\beta}}+5e^{\hat{\beta}}+q-6}K_{17} - \frac{8e^{2\hat{\beta}}}{4e^{2\hat{\beta}}+q-4}K_{18} - \frac{6e^{2\hat{\beta}}+2e^{\hat{\beta}}}{3e^{2\hat{\beta}}+2e^{\hat{\beta}}+q-5}K_{19}$$
$$- \frac{4e^{2\hat{\beta}}+4e^{\hat{\beta}}}{2e^{2\hat{\beta}}+4e^{\hat{\beta}}+q-6}K_{20} - \frac{2e^{2\hat{\beta}}+6e^{\hat{\beta}}}{e^{2\hat{\beta}}+6e^{\hat{\beta}}+q-7}K_{21} - \frac{8e^{\hat{\beta}}}{8e^{\hat{\beta}}+q-8}K_{22} = 0 \qquad (10)$$

MPL equation for q-state Potts model

The maximum pseudo-likelihood equation is numerically solved by using the Brent's method, a root-finding algorithm that does not require the computation (not even the existence) of derivatives or analytical gradients.

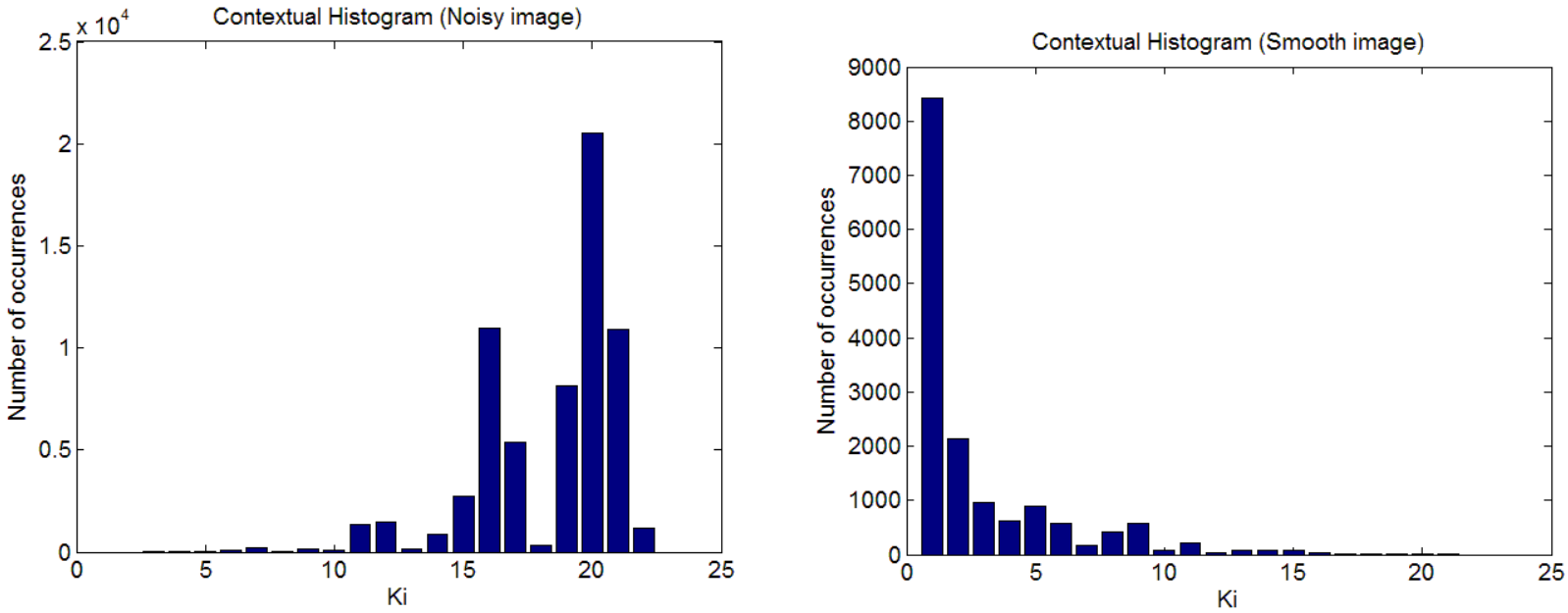

Figure : Comparison between the distribution of contextual configuration patterns for both smooth and noisy images (k_0 stands for total agreement and k_{22} for zero agreement).

Markov Chain Monte Carlo Algorithms

Algorithm 1: Metropolis-Hastings

Input : An initial $S_x \times S_y$ random field configuration; The maximum number of iterations N_{iter}.

Output: An output of the desired random field model.

Define a random field model and its parameters.
Initialize a random field configuration by randomly choosing each element of the system. Name this configuration x.

while $n \leq N_{iter}$ **do**
  for $i \leftarrow 1$ **to** S_x **do**
    for $j \leftarrow 1$ **to** S_y **do**
      Choose a new random value g and set $y_{ij} = g$.
      Let $p = min\left\{1, \frac{P(X=y)}{P(X=x)}\right\}$.
      Replace x by y with probability p.
    end
  end
end

Markov Chain Monte Carlo Algorithms

Algorithm 2: Gibbs Sampler

Input : An initial $S_x \times S_y$ random field configuration; The maximum number of iterations N_{iter}.

Output: An output of the desired random field model.

Define a random field model and its parameters.
Initialize a random field configuration by randomly choosing each element of the system. Name this configuration x.

while $n \leq N_{iter}$ **do**
  for $i \leftarrow 1$ **to** S_x **do**
    for $j \leftarrow 1$ **to** S_y **do**
      Compute the set of probabilities $\{p_g\}$, $\forall g \in G$, *where* $p_g = p\left(x_{ij} = g | \eta_{ij}\right)$ *with* η_{ij} *denoting the neighborhood of* x_{ij}.
      Assign the label g *to* x_{ij} *with probability* p_g.
    end
  end
end

Information Geometry

Information geometry is a branch of mathematics that provides a robust and geometrical treatment to most parametric statistical models

When we analyse isolated random variables (independent), the scenario is extensively known, with the underlying statistical manifolds being completely characterized

However, little is known about the scenario in which we have several variables interacting with each other (random fields)

Fisher information and Information Geometry

The concept of Fisher information has been present in an ubiquitous manner throughout mathematical statistics, playing an important role in several applications, from numerical estimation methods based on the Newton-Raphson iteration to the definition of lower bounds in unbiased estimation (Cramer-Rao lower bound).

With the development of information geometry, another fundamental role of Fisher information in statistical models has been discovered: it defines intrinsic geometric properties of the parametric space of a model, by characterizing the metric tensor of the respective manifold.

Fisher information and Information Geometry

Fisher information matrix is the metric tensor of the Riemannian manifold that defines the underlying parametric space of a random field

The metric tensor makes it possible to express the square of an infinitesimal displacement in the manifold, ds^2, as a function of an infinitesimal displacement of the parameters

$$ds^2 = \begin{bmatrix} d\theta & d\beta \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} d\theta \\ d\beta \end{bmatrix} = Ad\theta^2 + 2Bd\theta d\beta + Cd\beta^2 \qquad (11)$$

where the matrix of coefficients A, B, e C is the metric tensor.

Fisher information

In the context of information theory, Fisher information measures the amount of information a random sample conveys about an unknown parameter

Definition

Let $p(X;\vec{\theta})$ be a probability density function where $\vec{\theta} = (\theta_1, \ldots, \theta_n) \in \Theta$ is the parametric vector. The Fisher information matrix, which is the natural Riemannian metric of the parametric space, is defined as:

$$\left\{ I(\vec{\theta}) \right\}_{ij} = E\left[\left(\frac{\partial}{\partial \theta_i} log\ p(X;\vec{\theta}) \right) \left(\frac{\partial}{\partial \theta_j} log\ p(X;\vec{\theta}) \right) \right], i,j = 1, \ldots, n \tag{12}$$

Information Equality

In some cases, it is possible to compute the expected Fisher information matrix of a model by two different but equivalent ways (since the integration and differentiation operators can be interchangeable), defining the condition known as the information equality:

$$E\left[\left(\frac{\partial}{\partial\theta_i} log\ p(X;\vec{\theta})\right)\left(\frac{\partial}{\partial\theta_j} log\ p(X;\vec{\theta})\right)\right] = -E\left[\frac{\partial^2}{\partial\theta_i\partial\theta_j} log\ p(X;\vec{\theta})\right] \tag{13}$$

However, in random field models information equality is not a natural condition due to the inverse temperature parameter. As β drifts apart from zero, we need to consider 2 types of Fisher information. In this investigation we are concerned only with the Fisher information regarding the inverse temperature parameter.

Fisher information in Gaussian random fields

Let Σ_p be the covariance matrix of the random vectors $\vec{p}_i, i = 1, 2, \ldots, n$, obtained by lexicographic ordering the local configuration patterns $x_i \cup \eta_i$.

In this work, we choose a second-order neighborhood system, making each local configuration pattern a 3×3 patch. Thus, since each vector $\vec{p}_i$ has 9 dimensions, the resulting covariance matrix Σ_p is 9×9.

Let Σ_p^- be the sub-matrix of dimensions 8×8 obtained by removing the central row and central column of Σ_p (these elements are the covariances between x_i and each one of its neighbors $x_j \in \eta_i$).

Also, let $\vec{\rho}$ be the vector of dimensions 8×1 formed by all the elements of the central row of Σ_p, excluding the middle one (which is the variance of x_i actually).

Decomposing the covariance matrix

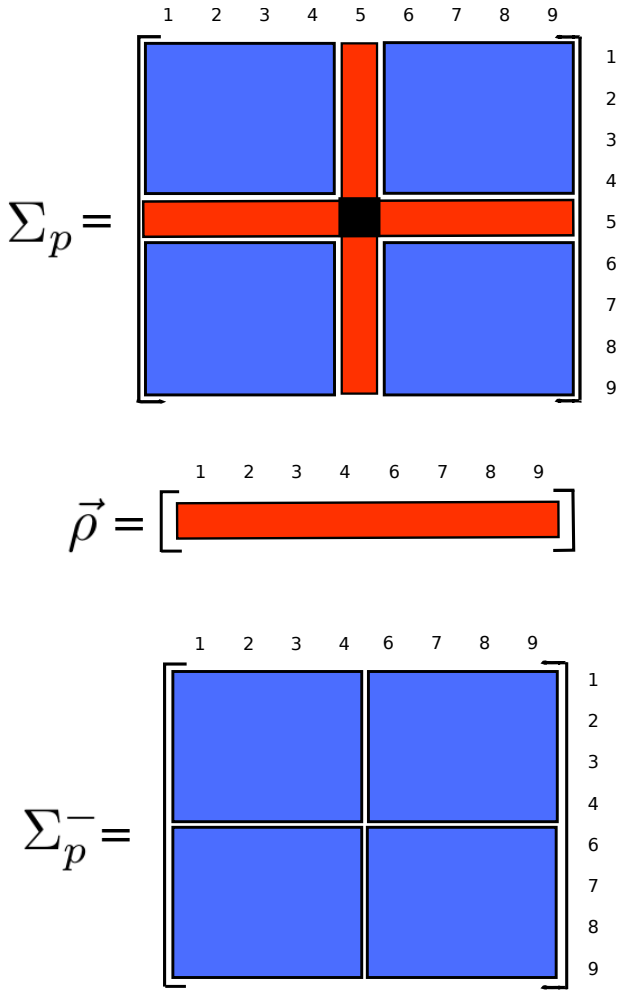

Figure : Decomposing the covariance matrix Σ_p into Σ_p^- and $\vec{\rho}$ on a second-order neighborhood system ($K = 8$). By expressing both Φ_β and Ψ_β in terms of Kronocker products, it is possible to compute Fisher information in a efficient way during computational simulations.

Fisher information in Gaussian random fields

Definition

Let an isotropic pairwise GMRF be defined on a lattice $S = \{s_1, s_2, \ldots, s_n\}$ *with a neighborhood system* η_i *of size* K *(usual choices for* K *are even values: 4, 8, 12, 20 or 24). Assuming that* $\mathbf{X^{(t)}} = \{x_1^{(t)}, x_2^{(t)}, \ldots, x_n^{(t)}\}$ *denotes the global configuration of the system at time* t*, the type-I expected Fisher information* Φ_β *for* $\mathbf{X^{(t)}}$ *is:*

$$\Phi_\beta = \frac{1}{\sigma^2}\left\|\Sigma_p^-\right\|_+ + \frac{1}{\sigma^4}\left[2\left\|\vec{\rho}\otimes\vec{\rho}^T\right\|_+ - 6\beta\left\|\vec{\rho}^T\otimes\Sigma_p^-\right\|_+ + 3\beta^2\left\|\Sigma_p^-\otimes\Sigma_p^-\right\|_+\right. \tag{14}$$

where $\|A\|_+$ denotes the summation of all the entries of the matrix A (not to be confused with a matrix norm) and $\otimes$ denotes the Kronecker (tensor) product. Similarly, it is possible to define Ψ_β using a matrix-vector notation and tensor products.

Fisher information in Gaussian random fields

Definition

Let an isotropic pairwise GMRF be defined on a lattice $S = \{s_1, s_2, \ldots, s_n\}$ *with a neighborhood system* η_i *of size* K *(usual choices for* K *are 4, 8, 12, 20 or 24). Assuming that* $\mathbf{X^{(t)}} = \{x_1^{(t)}, x_2^{(t)}, \ldots, x_n^{(t)}\}$ *denotes the global configuration of the system at time* t*, the type-II expected Fisher information* Ψ_β *for* $\mathbf{X^{(t)}}$ *is:*

$$\Psi_\beta = \frac{1}{\sigma^2} \left\| \Sigma_p^- \right\|_+ \tag{15}$$

Entropy in Gaussian random fields

Definition

Let an isotropic pairwise GMRF be defined on a lattice $S = \{s_1, s_2, \ldots, s_n\}$ *with a neighborhood system* η_i. *Assuming that* $\mathbf{X^{(t)}} = \{x_1^{(t)}, x_2^{(t)}, \ldots, x_n^{(t)}\}$ *denotes the global configuration of the system at time* t, *then the entropy* H_β *for this state* $\mathbf{X^{(t)}}$ *is given by:*

$$H_\beta = -E\left[log\ p\left(x_i|\eta_i, \vec{\theta}\right)\right] = \frac{1}{2}\left[log\left(2\pi\sigma^2\right) + 1\right] \tag{16}$$
$$- \frac{1}{\sigma^2}\left[\beta \sum_{j\in\eta_i} \sigma_{ij} - \frac{\beta^2}{2} \sum_{j\in\eta_i} \sum_{k\in\eta_i} \sigma_{jk}\right]$$

Fisher information in q-state Potts model

Definition

Let an isotropic pairwise q-state Potts model be defined on a lattice $S = \{s_1, s_2, \ldots, s_n\}$ *with a neighborhood system* η_i *of size* K*. Assuming that* $\mathbf{X^{(t)}} = \{x_1^{(t)}, x_2^{(t)}, \ldots, x_n^{(t)}\}$ *denotes the global configuration of the system at time* t*, the type-I and type-II observed Fisher information* ϕ_β *and* ψ_β *for* $\mathbf{X^{(t)}}$ *is:*

$$\phi_\beta = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\left\|\left(\vec{v}_i \odot \vec{w}_i\right) \otimes \left(\vec{v}_i \odot \vec{w}_i\right)^T\right\|_+}{\left\|\vec{w}_i \otimes \vec{w}_i^T\right\|_+}\right] \tag{17}$$

$$\psi_\beta = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\left\|\Lambda_i \odot \left(\vec{w}_i \otimes \vec{w}_i^T\right)\right\|_+}{\left\|\vec{w}_i \otimes \vec{w}_i^T\right\|_+}\right] \tag{18}$$

where $\vec{v}_i$, $\vec{w}_i$ and $\Lambda_i = A_i \odot B_i$ are defined according to the next equations.

Fisher information in q-state Potts model

$$\vec{v}_i = \begin{bmatrix} U_i(x_i) - U_i(1) \\ U_i(x_i) - U_i(2) \\ \vdots \\ U_i(x_i) - U_i(q) \end{bmatrix} \tag{19}$$

$$\vec{w}_i = \begin{bmatrix} e^{\beta U_i(1)} \\ e^{\beta U_i(2)} \\ \vdots \\ e^{\beta U_i(q)} \end{bmatrix} \tag{20}$$

Fisher information in q-state Potts model

$$A_i = \begin{bmatrix} U_i(1) & U_i(1) & U_i(1) & \cdots & U_i(1) \\ U_i(2) & U_i(2) & U_i(2) & \cdots & U_i(2) \\ \vdots & & & & \\ U_i(q) & U_i(q) & U_i(q) & \cdots & U_i(q) \end{bmatrix} \tag{21}$$

$$B_i = \begin{bmatrix} 0 & U_i(1) - U_i(2) & U_i(1) - U_i(3) & \cdots & U_i(1) - U_i(q) \\ U_i(2) - U_i(1) & 0 & U_i(2) - U_i(3) & \cdots & U_i(2) - U_i(q) \\ \vdots & & & & \\ U_i(q) - U_i(1) & 0 & U_q(2) - U_i(3) & \cdots & 0 \end{bmatrix} \tag{22}$$

with $\Lambda_i = A_i \odot B_i$, where $\odot$ denotes the Hadamard product (point-wise) between two matrices

Entropy in q-state Potts model

Definition

Let an isotropic pairwise q-state Potts model be defined on a lattice $S = \{s_1, s_2, \ldots, s_n\}$ *with a neighborhood system* η_i. *Assuming that* $\mathbf{X^{(t)}} = \{x_1^{(t)}, x_2^{(t)}, \ldots, x_n^{(t)}\}$ *denotes the global configuration of the system at time* t, *then the entropy* H_β *for this state* $\mathbf{X^{(t)}}$ *is given by:*

$$H_\beta = -E\left[log\ p\left(x_i|\eta_i, \vec{\theta}\right)\right] \approx \frac{1}{n}\sum_{i=1}^{n}\left[log\left\|\vec{w}_i\right\|_+ - \beta U_i(x_i)\right] \tag{23}$$

Fisher curves

Definition

Let an isotropic pairwise GMRF model be defined on a lattice $S = \{s_1, s_2, \ldots, s_n\}$ *with a neighborhood system* η_i *and* $\mathbf{X}^{(\beta_1)}, \mathbf{X}^{(\beta_2)}, \ldots, \mathbf{X}^{(\beta_n)}$ *be a sequence of outcomes (global configurations) produced by different values of* β_i *(inverse temperature parameters) for which* $A = \beta_{MIN} = \beta_1 < \beta_2 < \cdots < \beta_n = \beta_{MAX} = B$. *The Fisher curve from* A *to* B *is defined as the parametric curve* $\vec{F} : \Re \rightarrow \Re^3$ *that maps each configuration* $\mathbf{X}^{(\beta_i)}$ *to a point* $(\Phi_\beta, \Psi_\beta, H_\beta)$ *in the information space:*

$$\vec{F}_A^B(\beta) = (\Phi_\beta, \Psi_\beta, H_\beta) \qquad \beta = A, \ldots, B \tag{24}$$

where Φ_β, Ψ_β and H_β denote the type-I Fisher information, type-II Fisher information and entropy, respectively.

Fisher curves

The motivation behind the Fisher curve is the development of a computational tool for the study and characterization of random fields.

The Fisher curve of a system is the parametric curve embedded in this information-theoretic space obtained by varying the inverse temperature parameter β from an initial value β_I to a final value β_F.

The resulting curve provides a geometrical interpretation about how the random field evolves from a lower entropy configuration A to a higher entropy configuration B (or vice-versa), since the Fisher information plays an important role in providing a natural metric to the Riemannian manifold of a statistical model.

We will call the path from a global system configuration A to a global system configuration B as the *Fisher curve* (from A to B) of the system, denoted by $\vec{F}_A^B(\beta)$.

Computational Simulations

Our main objective is to measure Φ_β, Ψ_β and H_β along a MCMC simulation in which the inverse temperature parameter β is controlled to guide the global system behavior.

By sensing a component of the metric tensor (Fisher information) at each point, we are trying to capture part of the deformation in the geometric structure of the manifold defined by the random field's parametric space throughout the process.

Fisher curves in Gaussian random fields

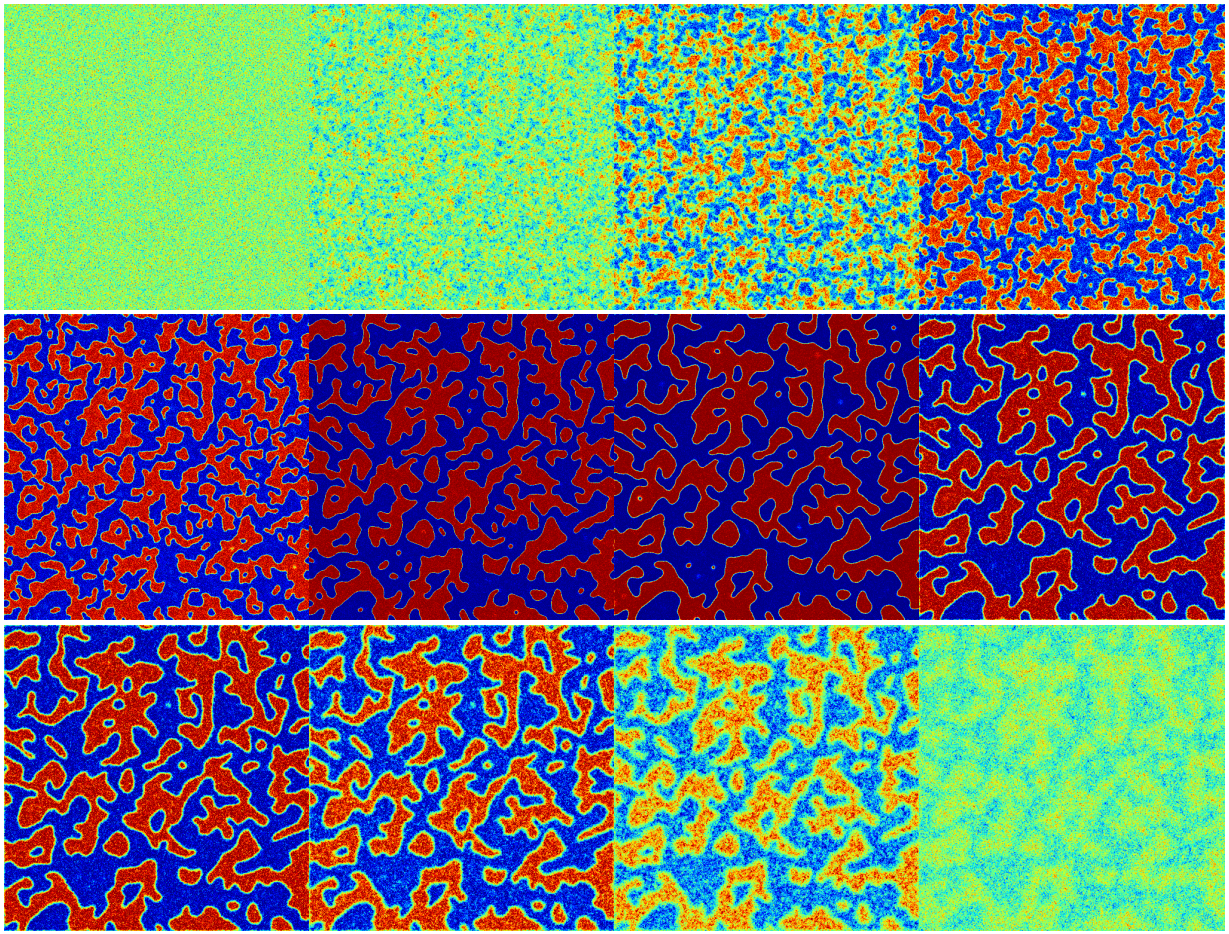

Figure : **Gaussian random field dynamics along a Markov Chain Monte Carlo (MCMC) simulation.** Evolution of the random field as the inverse temperature parameter β is first increased from zero to 0.5 and then decreased from 0.5 to zero.

Fisher curves in Gaussian random fields

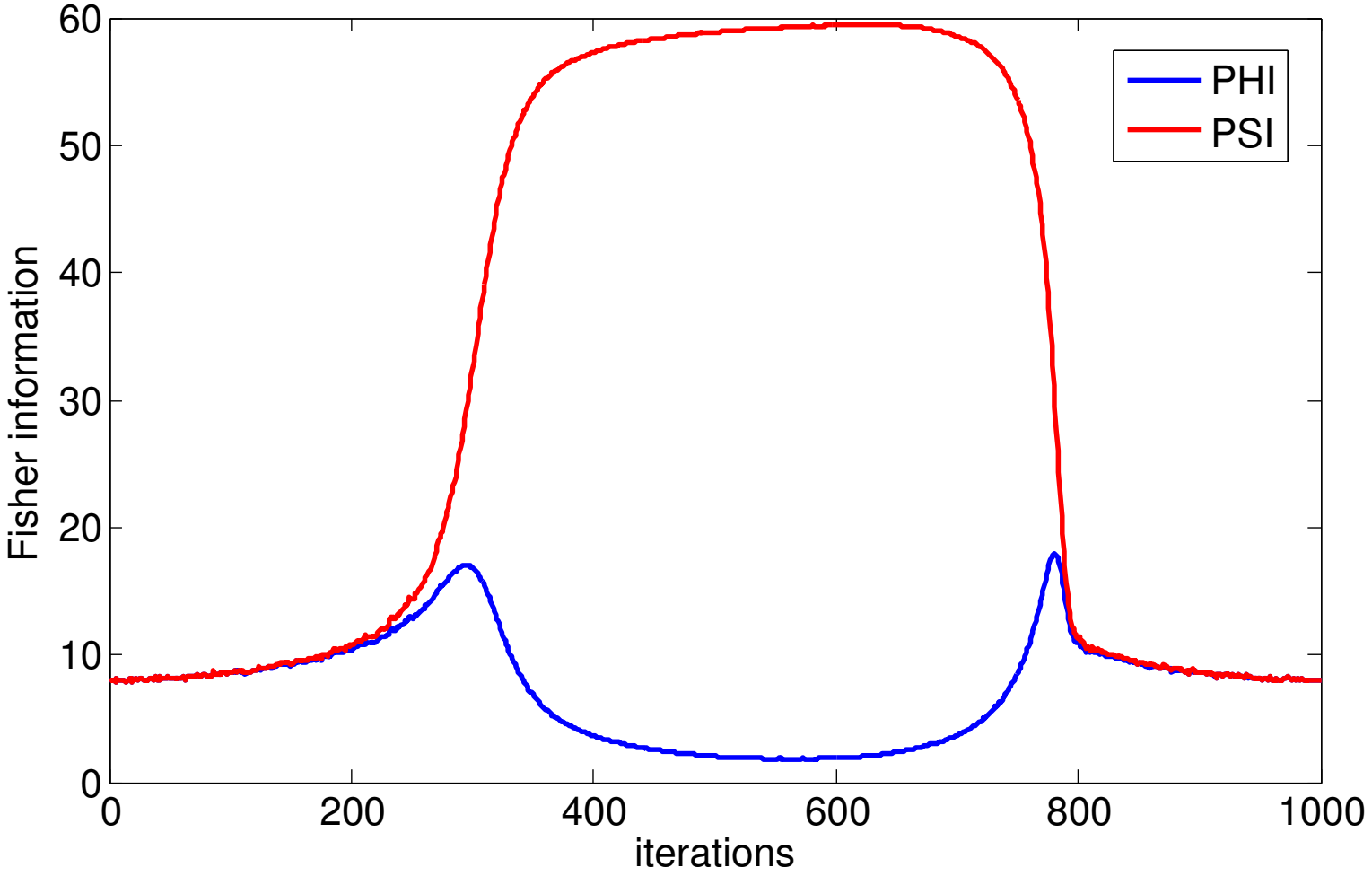

Figure : Fisher information in the isotropic pairwise GMRF model along the MCMC simulation. When the temperature is infinity ($\beta = 0$), the information equality prevails, however, for larger values of β, Φ_β and Ψ_β diverge.

Fisher curves in Gaussian random fields

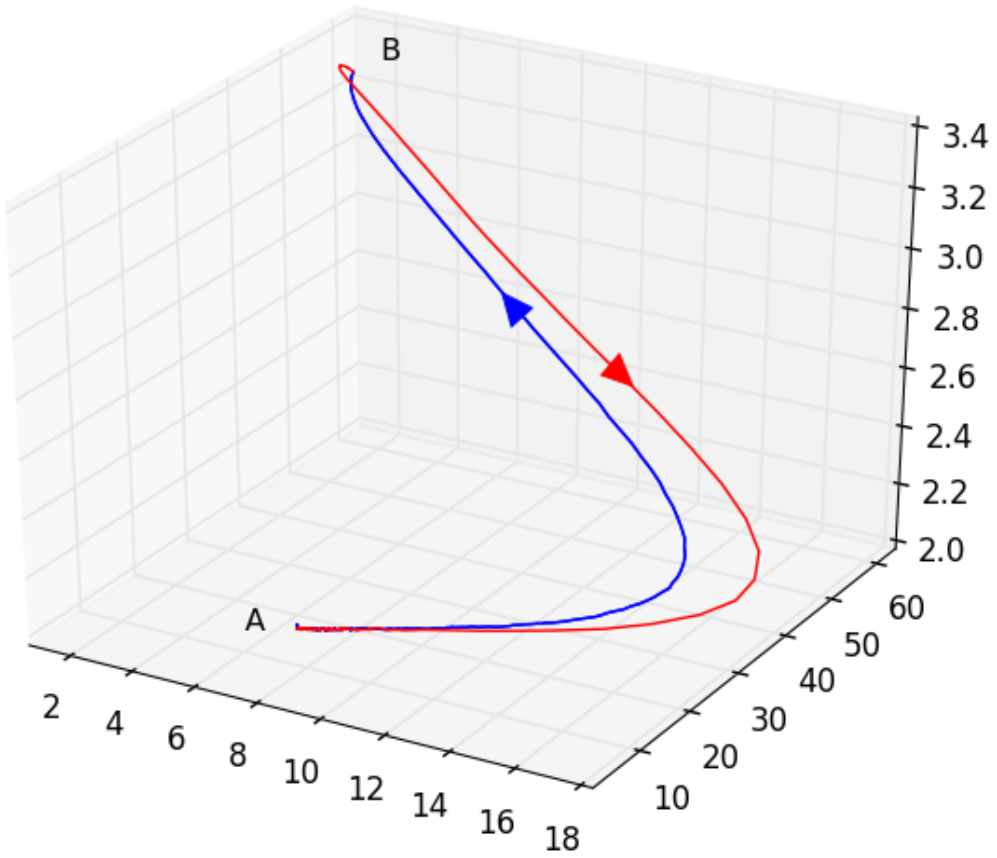

Figure : **Fisher curve of the random field.** The results show that moving along different entropic states causes the emergence of a natural orientation in terms of information (an arrow of time). This behavior resembles the conceptual idea of the phenomenon known as hysteresis.

Fisher curves in q-state Potts model

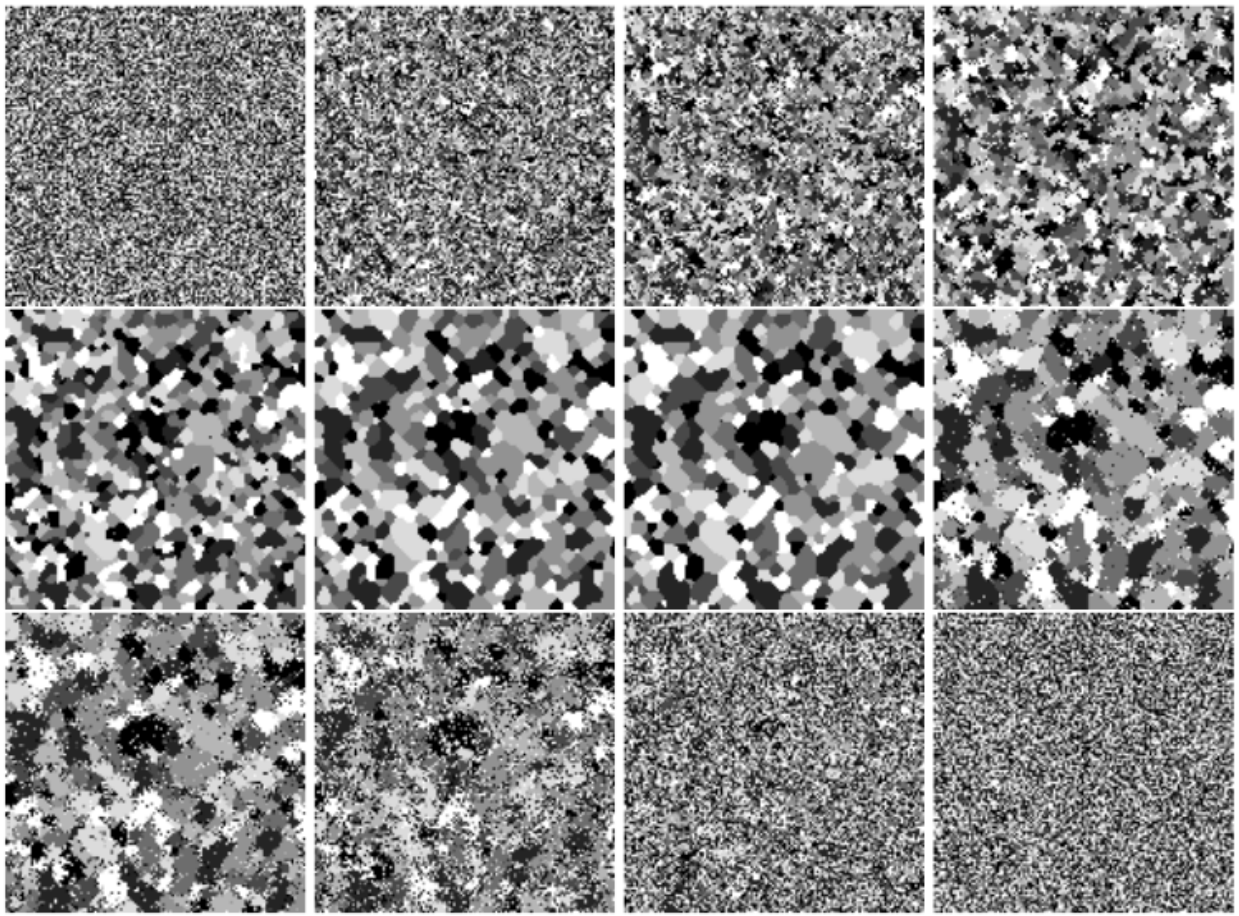

Figure : **8-state Potts random field dynamics along a Markov Chain Monte Carlo (MCMC) simulation.** Evolution of the random field as the inverse temperature parameter β is first increased then decreased back to zero.

Fisher curves in q-state Potts model

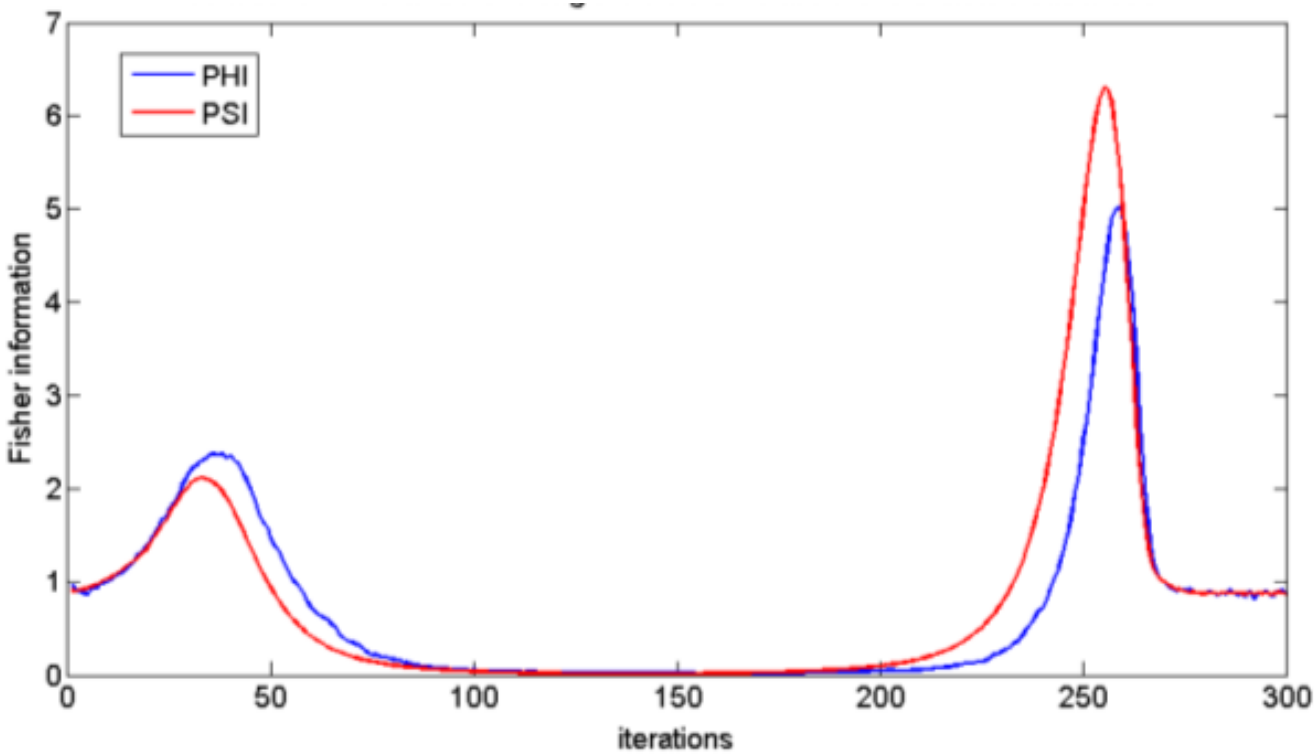

Figure : **Fisher information in a 8-state Potts random field.** Evolution of Φ_β and Ψ_β as the inverse temperature parameter β is first increased then decreased back to zero. Note that there is an inversion in the dominance between Φ_β and Ψ_β.

Fisher curves in q-state Potts model

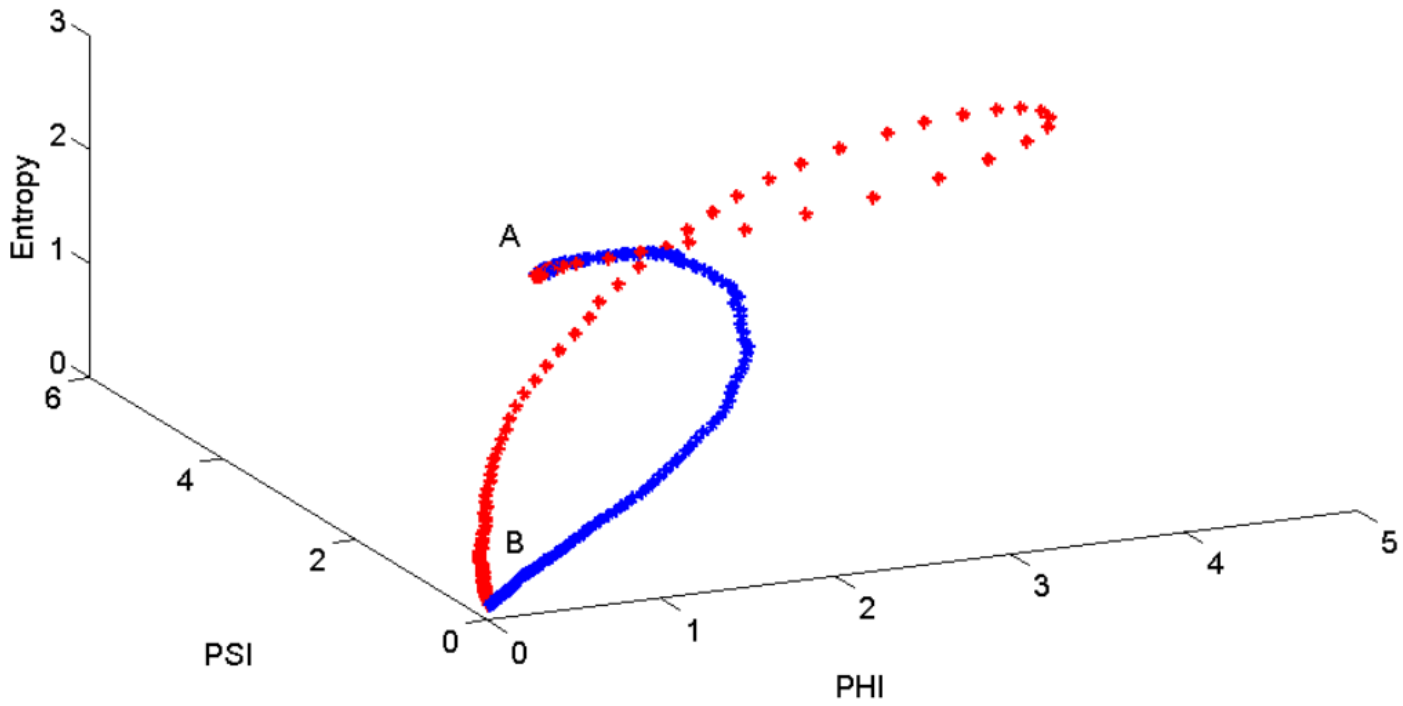

Figure : **3-D Fisher curve in a 8-state Potts random field.** The trajectory in the information space from a higher entropy state to lower entropy state and back indicates a unique orientation in the evolution of random field dynamics. This natural orientation emerges when the inverse temperature parameter significantly deviates from zero, suggesting the emergence of an arrow of time in such complex systems. Once again, note that the Fisher curve resembles a mathematical model of hysteresis.

Conclusions and Final Remarks

In this overview, we addressed the problem of characterizing the Fisher curves of random field models as mathematical tools for the study of the evolution of complex systems.

To investigate the dynamics of such systems, we performed computational simulations in which the inverse temperature parameter is controlled to guide the system behavior throughout different entropic states.

Basically, the Fisher curve of the system provides a geometrical tool for the analysis of random fields by showing how different entropic states are "linked" in terms of Fisher information, which is, by definition, the metric tensor of the underlying random field model parametric space. In other words, when the random field moves along different entropic states, its parametric space is actually being deformed by changes that happen in Fisher information matrix (the metric tensor).

Conclusions and Final Remarks

The main conclusion of this investigation can be summarized as: in random fields, moving towards lower entropy states is different form moving towards higher entropy states, since the Fisher curves are not the same. This asymmetry induces a natural orientation to the process of taking the random field from an initial state A to a final state B and back, which is basically the direction pointed by an arrow of time, since the only way to move in the opposite direction is by running the simulations backwards. In this context, the Fisher curve resembles a mathematical model of hysteresis.