On the Conformal Invariant Cosmology

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Be'er Sheva, Israel — Symmetry — 2021

1. Local conformal transformation

   Metric tensor
   $ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \qquad \mu, \nu = 0, 1, 2, 3$
   $ds^2 = \Omega^2(x) d\hat{s}^2$
   $g_{\mu\nu} = \Omega^2(x) \hat{g}_{\mu\nu}$
   $\sqrt{-g} = \Omega^4(x) \sqrt{-\hat{g}}$

2. Cosmological principle $=$ homogeneous and isotropic Universe
   Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \right)$$

$(+, -, -, -), \quad k = \pm 1, \quad c = 1$

$a(t)$ — scale factor
t — cosmological time
$k = +1$ — closed universe
$k = -1$ — open universe
$k = 0$ — spatially flat universe

Motivation

Creation of the universe from "nothing"
(quantum tunneling)
A.A. Friedmann (1923)
A.V. Vilenkin (1984)
Ya.B. Zel'dovich
S.W. Hawking

Creation probability

$$P \sim e^{-S_{\rm tot}}$$

$S_{\rm tot}$ — total action integral under the potential barrier

$$S_{\rm tot} = S_{\rm grav} + S_{\rm matter}$$

Universe is created being empty $\rightarrow$ $S_{\rm matter} = 0$ $\Rightarrow$
The smaller $S_{\rm grav}$, the better.
The more symmetry, the smaller $S_{\rm grav}$

Some of differential geometry

Main objects: scalars, vectors, tensors

Metric tensor $g_{\mu\nu}$

Connections $\Gamma^{\lambda}_{\mu\nu}$

Covariant differentiation (Leibniz rule)

$$\nabla_\lambda l^\mu = l^\mu_{,\lambda} + \Gamma^\mu_{\lambda\sigma} l^\sigma$$

$$\nabla_\lambda l_\mu = l_{\mu,\lambda} - \Gamma^\sigma_{\lambda\mu} l_\sigma$$

(“,” = partial derivative)

Curvature tensor $R^\mu{}_{\nu\lambda\sigma}$

$$R^\mu{}_{\nu\lambda\sigma} = \frac{\partial \Gamma^\mu_{\nu\sigma}}{\partial x^\lambda} - \frac{\partial \Gamma^\mu_{\nu\lambda}}{\partial x^\sigma} + \Gamma^\mu_{\varkappa\lambda}\Gamma^\varkappa_{\nu\sigma} - \Gamma^\mu_{\varkappa\sigma}\Gamma^\varkappa_{\nu\lambda}$$

$$(R^\mu{}_{\nu\lambda\sigma} = -R^\mu{}_{\nu\sigma\lambda})$$

Ricci tensor $R_{\mu\nu}$

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$$

Curvature scalar

$$R = R^\lambda_\lambda$$

Some of differential geometry II

The connections $\Gamma^{\lambda}_{\mu\nu}$ are completely defined if one knows the Christoffel symbols $C^{\lambda}_{\mu\nu}$, the torsion tensor $T^{\lambda}_{\mu\nu}$ and the nonmetricity tensor $Q^{\lambda}_{\mu\nu}$

$$\Gamma^{\lambda}_{\mu\nu} = C^{\lambda}_{\mu\nu} + K^{\lambda}_{\mu\nu} + L^{\lambda}_{\mu\nu}$$

$$C^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}(g_{\kappa\mu,\nu} + g_{\kappa\nu,\mu} - g_{\mu\kappa,\nu}) \quad \Rightarrow \quad C^{\lambda}_{\mu\nu} = C^{\lambda}_{\nu\mu}$$

$$T^{\lambda}_{\mu\nu} = (\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu})$$

$$Q_{\lambda\mu\nu} = \nabla_{\lambda} g_{\mu\nu} \quad \Rightarrow \quad Q_{\lambda\mu\nu} = Q_{\lambda\nu\mu}$$

$$K^{\lambda}{}_{\mu\nu} = \frac{1}{2}(T^{\lambda}{}_{\mu\nu} - T_{\mu}{}^{\lambda}{}_{\nu} - T_{\nu}{}^{\lambda}{}_{\mu})$$

$$L^{\lambda}_{\mu\nu} = \frac{1}{2}(Q^{\lambda}{}_{\mu\nu} - Q_{\mu}{}^{\lambda}{}_{\nu} - Q_{\nu}{}^{\lambda}{}_{\mu})$$

Quadratic Gravity

We confine ourselves to the case of Quadratic gravity:

1. Most natural in constructing conformal invariant action integrals in 4-dim
2. appears in the trace anomaly formulas for one-loop quantum calculations:
   A.D. Sakharov's induced gravity (1966)
   L. Parker and S. Fulling
   Ya.B. Zel'dovich and A.A. Starobinsky
   A.A. Grib, S.G. Mamaev, V.M. Mostepanenko
   ...

$$S_2 = \int \mathcal{L}_2 \sqrt{-g}\, d^4x$$

$$\mathcal{L}_2 = \alpha_1 R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R^2 + \alpha_4 R + \alpha_5 \Lambda$$

Riemannian Geometry

$$T^{\lambda}_{\mu\nu} = 0, \quad Q_{\lambda\mu\nu} = 0, \quad \Rightarrow \quad \Gamma^{\lambda}_{\mu\nu} = C^{\lambda}_{\mu\nu}$$

Symmetries

$$R_{\mu\nu\lambda\sigma} = R_{\lambda\sigma\mu\nu} = -R_{\nu\mu\lambda\sigma} = -R_{\mu\nu\sigma\lambda}$$

$$R_{\mu\nu\lambda\sigma} + R_{\mu\sigma\nu\lambda} + R_{\mu\lambda\sigma\nu} = 0$$

$$R_{\mu\nu} = R_{\nu\mu}$$

Bianchi identities

$$R^{\mu}_{\nu\lambda\sigma;\kappa} + R^{\mu}_{\nu\kappa\lambda;\sigma} + R^{\mu}_{\nu\sigma\kappa;\lambda} = 0$$

$(;)$ = covariant derivative with $C^{\lambda}_{\mu\nu}$

Weyl tensor: completely traceless part of the Riemann curvature tensor (obeying all the above algebraic symmetries)

$$C_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\sigma} + \frac{1}{2}(R_{\mu\sigma}g_{\nu\lambda} - R_{\mu\lambda}g_{\nu\sigma} - R_{\nu\sigma}g_{\mu\lambda}$$

$$+R_{\nu\lambda}g_{\mu\sigma}) + \frac{1}{6}R(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda})$$

Conformal transformation

$$C^{\mu}{}_{\nu\lambda\sigma} = \hat{C}^{\mu}{}_{\nu\lambda\sigma} \quad \text{— invariant}$$

$$C^2 = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$$

$$C^2 = C_{\mu\nu\lambda\sigma}C^{\mu\nu\lambda\sigma} = g_{\mu\mu'}g^{\nu\nu'}g^{\lambda\lambda'}g^{\sigma\sigma'}R^{\mu}{}_{\nu\lambda\sigma}R^{\mu'}{}_{\nu'\lambda'\sigma'} \quad \Rightarrow$$

$$C^2 = g_{\mu\mu'}g^{\nu\nu'}g^{\lambda\lambda'}g^{\sigma\sigma'}\hat{C}^{\mu}{}_{\nu\lambda\sigma}\hat{C}^{\mu'}{}_{\nu'\lambda'\sigma'} =$$

$$= \frac{\Omega^2}{\Omega^2}\hat{C}^2 = \frac{1}{\Omega^4}\hat{C}^2 \quad \Rightarrow \quad C^2\sqrt{-g} = \hat{C}^2\sqrt{-\hat{g}}$$

Gauss-Bonnet term

$$GB = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

In 4-dim — full derivative $\Rightarrow$ does not contribute to the field equations

Quadratic gravity in Riemannian Geometry

Lagrangian $\mathcal{L}_2$ rewritten

$$\mathcal{L}_2 = \alpha_1 R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R^2 + \alpha_4 R + \alpha_5 \Lambda$$

$$= \alpha C^2 + \beta GB + \gamma R^2 + \alpha_4 R + \alpha_5 \Lambda$$

$$\begin{cases} \alpha = 2\alpha_1 + \frac{1}{2}\alpha_2 \\ \beta = -\alpha_1 - \frac{1}{2}\alpha_2 \\ \gamma = \frac{1}{3}(\alpha_1 + \alpha_2 + 3\alpha_3) \end{cases}$$

$\Rightarrow$ Only the C^2-term is suitable for construction the conformal invariant gravitational action integral in 4-dim

$$S_2 \quad \rightarrow \quad \alpha \int C^2 \sqrt{-g}\, d^4x$$

Field equations

$$B_{\mu\nu} = -\frac{1}{8\alpha} T_{\mu\nu}$$

$B_{\mu\nu}$ — Bach tensor

$$B^{\mu\nu} = C^{\mu\sigma\nu\lambda}{}_{;\lambda;\sigma} + \frac{1}{2} C^{\mu\sigma\nu\lambda} R_{\sigma\lambda}$$

$$B^{\mu\nu} = B^{\nu\mu}, \quad B^{\lambda}_{\lambda} = 0, \quad B^{\mu\nu}{}_{;\nu} = 0$$

Cosmology = homogeneity and isotropy
For the Robertson-Walker metric with any scale factor $a(t)$ one has

$$C^{\mu}{}_{\nu\lambda\sigma} \equiv 0 \quad \Rightarrow \quad B^{\mu\nu} = 0 \quad \Rightarrow \quad T^{\mu\nu} = 0$$

> There are no conformal invariant solutions with matter fields in Riemannian geometry!

Note: if one forgets for a moment about the conformal invariance, then for the cosmology $\mathcal{L}_2 = \gamma R^2 + \alpha_4 R + \alpha_5 \Lambda$, i. e., the Starobinsky inflationary model.

Hermann Weyl — 1919

Unification of gravitational and electromagnetic forces
Main idea — Maxwell equations are locally conformal invariant $\rightarrow$ gravitation should also be conformal invariant.
He incorporated an electromagnetic potential A_μ into geometry by the relation

$$\nabla_\lambda g_{\mu\nu} = A_\lambda g_{\mu\nu}$$

and demanded the local conformal invariance for the resulting theory. Under the conformal transformation

$$g_{\mu\nu} = \Omega^2(x)\hat{g}_{\mu\nu} \quad \rightarrow \quad A_\mu = \hat{A}_\mu + 2\frac{\Omega_{,\mu}}{\Omega},$$

i. e., A_μ is the gauge field mediating the conformal factor
— great discovery!

Criticism by A. Einstein: “second time” problem

Most crucial — dependence of the time on the history.

The theory was abandoned.

We will not identify the 1-form A_μ with the electromagnetic potential and consider the Weyl geometry as the conformal invariant gravitational vector-tensor theory.

Weyl geometry

$$T^{\alpha}_{\mu\nu} = 0, \quad Q_{\alpha\mu\nu} = A_\alpha g_{\mu\nu} \quad \Rightarrow \quad \Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$$

$$\Gamma^\lambda_{\mu\nu} = C^\lambda_{\mu\nu} + W^\lambda_{\mu\nu}$$

$$C^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}(g_{\kappa\mu,\nu} + g_{\kappa\nu,\mu} - g_{\mu\nu,\kappa})$$

$$W^\lambda_{\mu\nu} = -\frac{1}{2}(A_\mu \delta^\lambda_\nu + A_\nu \delta^\lambda_\mu - A^\lambda g_{\mu\nu})$$

Conformal transformation

$$g_{\mu\nu} = \Omega^2(x) \hat{g}_{\mu\nu}$$

$$A_\mu = \hat{A}_\mu + 2\frac{\Omega_{,\mu}}{\Omega} \quad \Leftrightarrow \quad \Gamma^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\mu\nu} \quad \Rightarrow$$

$$R^\mu{}_{\nu\lambda\sigma}[\Gamma] = \hat{R}^\mu{}_{\nu\lambda\sigma}[\hat{\Gamma}]$$

$$R_{\mu\nu}[\Gamma] = \hat{R}_{\mu\nu}[\hat{\Gamma}]$$

Weyl geometry II

Also

$$R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}\sqrt{-g} = \hat{R}_{\mu\nu\lambda\sigma}\hat{R}^{\mu\nu\lambda\sigma}\sqrt{-\hat{g}}$$

$$R_{\mu\nu}R^{\mu\nu}\sqrt{-g} = \hat{R}_{\mu\nu}\hat{R}^{\mu\nu}\sqrt{-\hat{g}}$$

$$R^2\sqrt{-g} = \hat{R}^2\sqrt{-\hat{g}}$$

and, surely

$$F_{\mu\nu}F^{\mu\nu}\sqrt{-g} = \hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\sqrt{-\hat{g}}$$

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = A_{\nu;\mu} - A_{\mu;\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$

General Quadratic Gravity in Weyl Geometry

$$S_{\mathrm{W}} = \int \mathcal{L}_{\mathrm{W}}\sqrt{-g}\, d^4x$$

$$\mathcal{L}_{\mathrm{W}} = \alpha_1 R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 R^2 + \alpha_4 F_{\mu\nu}F^{\mu\nu}$$

Total action integral

$$S_{\mathrm{tot}} = S_{\mathrm{W}} + S_{\mathrm{m}}$$

Important! The action integral for the matter fields, S_{m}, need not be conformal invariant, but the variation δS_{m}, does.
By definition

$$\delta S_{\mathrm{m}} \stackrel{\mathrm{def}}{=} -\frac{1}{2}\int T^{\mu\nu}(\delta g_{\mu\nu})\sqrt{-g}\,d^4x - \int G^{\mu}(\delta A_{\mu})\sqrt{-g}\,d^4x$$

$$+\int \frac{\partial \mathcal{L}_{\mathrm{W}}}{\partial \Psi}(\delta\Psi)\sqrt{-g}\,d^4x = 0$$

Ψ — collective dynamical variable describing the matter fields

$$\frac{\delta S_{\mathrm{m}}}{\delta \Psi} = 0 \quad \Rightarrow \quad \frac{\delta S_{\mathrm{m}}}{\delta \Omega} = 0$$

Total action integral II

$$\delta g_{\mu\nu} = \frac{2}{\Omega} g_{\mu\nu}(\delta\Omega) + \Omega^2(\delta\hat{g}_{\mu\nu})$$

$$\delta A_\mu = (\delta A_\mu) + 2(\delta(\log\Omega)_{,\mu}$$

$$\Rightarrow \quad -\int T^{\mu\nu} g_{\mu\nu}(\frac{\delta\Omega}{\Omega})\sqrt{-g}\, d^4x - 2\int G^\mu(\delta(\log\Omega)_{,\mu})\sqrt{-g}\, d^4x = 0$$

$$\Rightarrow \quad 2G^\mu_{;\mu} = Trace T^{\mu\nu}$$

This ensures the conformal invariance of the field equations and is supplementary.

Let us call it "the selfconsistent condition".

$$G^\mu = ? \quad G^\mu \stackrel{\mathrm{def}}{=} -\frac{\delta S_{\mathrm{m}}}{\delta A_\mu}$$

How could it be?

Perfect fluid in Riemannian geometry

Eulerian description

J.R. Ray (1972)

Action integral

$$S_{\mathrm{m}} = -\int \varepsilon(X,n)\sqrt{-g}\,d^4x + \int \lambda_0(u_\mu u^\mu - 1)\sqrt{-g}\,d^4x + \int \lambda_1 (nu^\mu)_{;\mu}\sqrt{-g}\,d^4x + \int \lambda_2 X_{,\mu}u^\mu\sqrt{-g}\,d^4x$$

Dynamical variables: $n(x)$ — invariant particle number density, $u^\mu(x)$ — four velocity vector, $X(x)$ — auxiliary variable, $\varepsilon(X,n)$ — invariant energy density

Equations of motion:

$$\delta n: \qquad -\frac{\partial\varepsilon}{\partial n} - \lambda_{1,\mu}u^\mu = 0$$

$$\delta u^\mu: \qquad 2\lambda_0 u^\mu - n\lambda_{1,\mu} + \lambda_2 X_{,\mu} = 0$$

$$\delta X: \qquad -\frac{\partial\varepsilon}{\partial X} - (\lambda_2 u^\mu)_{;\mu} = 0$$

Perfect fluid in Riemannian geometry II

$+$ constraints:

$$\lambda_0 : \qquad u^\mu u_\mu - 1 = 0$$

$$\lambda_1 : \qquad (nu^\mu)_{;\mu} = 0$$

$$\lambda_2 : \qquad X_{,\mu} u^\mu = 0$$ — numbering of the trajectories

Energy-momentum tensor $T^{\mu\nu}$:

$$T^{\mu\nu} = \varepsilon g^{\mu\nu} - 2\lambda_0 u^\mu u^\nu + n\lambda_{1,\sigma} u^\sigma g^{\mu\nu}$$

Hydrodynamical pressure p:

$$p = n\frac{\partial\varepsilon}{\partial n} - \varepsilon \quad \Rightarrow$$

$$(\varepsilon + p)u_\mu + n\lambda_{1,\mu} - \lambda_2 X_{,\mu} \quad \Rightarrow$$ Euler equation

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - pg^{\mu\nu}$$

Single particle in the given gravitational fields

Perfect fluid $\rightarrow$ dust $(p=0)$ $\rightarrow$ single particle
Dynamical variable: $x^\mu(\tau)$ — particle trajectory,
τ — the proper time

Riemannian geometry

The only invariant — the interval, s, along the trajectory $\Rightarrow$ (everybody knows)

$$S_{\rm part} = -m\int ds = -m\int\sqrt{g^{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau$$

m — particle mass

$\delta S_{\rm part} = 0$ $\Rightarrow$ shortest interval $=$ geodesics

$$\frac{d}{d\tau}\left(g^{\mu\lambda}(x)\frac{dx^\mu}{d\tau}\right) - \frac{1}{2}g_{\mu\nu,\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \qquad \left(\frac{dx^\mu}{d\tau} = u^\mu\right)$$

$$u_{\lambda,\nu}u^\nu - \frac{1}{2}g_{\mu\nu,\lambda}u^\mu u^\nu = 0 \quad \Rightarrow \quad u_{\lambda;\nu}u^\nu = 0 \quad = \text{ geodesics}$$

Weyl geometry

$A_\mu \quad \rightarrow$ yet another invariant (!)

$$B = A_\mu u^\mu \quad \Rightarrow$$

$$S_{\rm part} = \int f_1(B) ds + \int f_2(B) d\tau = \int \{ f_1(B)\sqrt{g_{\mu\nu}u^\mu u^\nu} + f_2(B) \} d\tau$$

Equations of motion

$$f_1 u_{\lambda;\sigma} u^\sigma = \left((f_1^{''} + f_1^{''}) A_\lambda - f_1^{'} u_\lambda \right) B_{,\mu} u^\mu + (f_1^{'} + f_2^{'}) F_{\lambda\mu} u^\mu$$

$$F_{\lambda\mu} = A_{\mu,\lambda} - A_{\lambda,\mu}$$

$$u_{\lambda;\sigma} u^\lambda = 0 \quad \Rightarrow$$

Either

$$(f_1^{''} + f_1^{''}) B - f_1^{'} = 0$$

or

$$B_{,\mu} u^\mu = 0$$

Perfect fluid in Weyl geometry

Before:

$$S_{\rm m} = -\int \varepsilon(X,n)\sqrt{-g}\,d^4x + \int \lambda_0(u_\mu u^\mu - 1)\sqrt{-g}\,d^4x + \int \lambda_1 (nu^\mu)_{;\mu}\sqrt{-g}\,d^4x + \int \lambda_2 X_{,\mu}u^\mu\sqrt{-g}\,d^4x$$

Recipe:

$$n \;\to\; \varphi(B)n \;\Rightarrow\; \varepsilon(X,n) \;\to\; \varepsilon(X,\varphi(B)n)$$

Besides,

$$(nu^\mu)_{;\mu}\sqrt{-g} = (nu^\mu\sqrt{-g})_{,\mu} = 0 \;\Rightarrow\; (\hat{n}\hat{u}^\mu\sqrt{-\hat{g}})_{,\mu} = 0$$

Particles can be just counted! (Physics)

Since $n = \dfrac{\hat{n}}{\Omega^4}$, $u^\mu = \dfrac{\hat{u}^\mu}{\Omega}$, $\sqrt{-g} = \Omega^4\sqrt{-\hat{g}} \;\Rightarrow$

$$(nu^\mu)_{;\mu} \neq 0 \quad \Rightarrow$$

$$(\varphi_1(B)nu^\mu)_{;\mu}nu^\mu)_{;\mu} = \Phi(B,n) \;\Rightarrow$$

Perfect fluid in Weyl geometry II

Now:

$$S_{\mathrm{m}} = -\int \varepsilon(X, \varphi(B)n)\sqrt{-g}\, d^4x + \int \lambda_0(u_\mu u^\mu - 1)\sqrt{-g}\, d^4x$$

$$+ \int \lambda_1 \left((\varphi_1(B)nu^\mu)_{;\mu} - \Phi(B,n)\right)\sqrt{-g}\, d^4x + \int \lambda_2 X_{,\mu} u^\mu \sqrt{-g}\, d^4x$$

Equations of motion

$$-\left(1 - B\left(\frac{\varphi'}{\varphi} - \frac{\varphi_1'}{\varphi_1}\right)\right)(\varepsilon + p)u_\mu + \left(\frac{\varphi'}{\varphi} - \frac{\varphi_1'}{\varphi_1}\right)(\varepsilon + p)A_\mu$$

$$+\lambda_1 \frac{\partial \Phi}{\partial B}(Bu_\mu - A_\mu) - \lambda_1\left(1 + B\frac{\varphi_1'}{\varphi_1}\right) n \frac{\partial \Phi}{\partial n} u_\mu$$

$$+\lambda_1 \frac{\varphi_1'}{\varphi_1} n \frac{\partial \Phi}{\partial n} A_\mu - n\varphi_1' \lambda_{1,\mu} + \lambda_2 X_{,\mu} = 0$$

$$-\frac{\partial \varepsilon}{\partial X} - (\lambda_1 u^\mu); \mu = 0$$

$$u^\mu u_\mu = 1, \quad (\varphi_1 n u^\mu)_{;\mu} = \Phi(B,n), \quad X_{,\mu} u^\mu = 0$$

Perfect fluid in Weyl geometry III

$$G^{\mu} = \left\{ \left(\frac{\varphi'}{\varphi} - \frac{\varphi_1'}{\varphi_1} \right) (\varepsilon + p) + \lambda_1 \frac{\partial \Phi}{\partial B} - \lambda_1 n \frac{\varphi_1'}{\varphi_1} \frac{\partial \Phi}{\partial n} \right\} u^{\mu}$$

$$T^{\mu\nu} = \left\{ \left(1 - B \left(\frac{\varphi'}{\varphi} - \frac{\varphi_1'}{\varphi_1} \right) \right) (\varepsilon + p) - \lambda_1 B \frac{\partial \Phi}{\partial B} \right.$$

$$\left. + \lambda_1 n \left(1 + B \frac{\varphi_1'}{\varphi_1} \right) \frac{\partial \Phi}{\partial n} \right\} u^{\mu} u^{\nu} + (-p + \lambda_1 \Phi - \lambda_1 n \frac{\partial \Phi}{\partial n}) g^{\mu\nu}$$

Cosmology

Homogeneity and isotropy $\Rightarrow$

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right)$$

$$= dt^2 - a^2(t)\gamma_{ij}dx^i dx^j = a^2(\eta)(d\eta^2 - \gamma_{ij}dx^i dx^j)$$

$$= \Omega^2(\eta)\hat{a}^2(\eta)(d\eta^2 - \gamma_{ij}dx^i dx^j)$$

$$T^\mu_\nu = (T^0_0, T^1_1 = T^2_2 = T^3_3)$$

Special gauge $=$ basic solution

$$A_\mu = 0$$

Field equations

$$-6\gamma\dot{R} = G^0$$

$$\begin{cases} -12\gamma\left\{\frac{\dot{a}}{a}\dot{R} + R\left(\frac{R}{12} + \frac{\dot{a}^2+k}{a^2}\right)\right\} = T_0^0 \\ -4\gamma\left\{\ddot{R} + 2\frac{\dot{a}}{a}\dot{R} - R\left(\frac{R}{12} + \frac{\dot{a}^2+k}{a^2}\right)\right\} = T_1^1 \end{cases}$$

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right)$$

$$2\frac{(G^0 a^3)^{\cdot}}{a^3} = T_0^0 + 3T_1^1 \quad \text{— selfconsistency condition}$$

$$\dot{T}_0^0 + 3\frac{\dot{a}}{a}(T_0^0 - T_1^1) = 0 \quad \text{— conservativity}$$

$$G^0 = -\left\{\left(\frac{\varphi'}{\varphi} - \frac{\varphi_1'}{\varphi_1}\right)(\varepsilon + p) + \lambda_1\frac{\varphi_1'}{\varphi_1}n\frac{\partial\Phi}{\partial n} + \lambda_1\frac{\partial\Phi}{\partial B}\right\}$$

$$T_0^0 = \varepsilon + \lambda_1\Phi$$

$$T_1^1 = -p - \lambda_1 n\frac{\partial\Phi}{\partial n} + \lambda_1\Phi$$

Field equations II

Equations of motion:

$$+(\varepsilon + p) + \lambda_1 n \frac{\partial \Phi}{\partial n} + \varphi_1 n \dot{\lambda}_1 = 0$$

$$\frac{(\varphi_1 n a^3)^{\cdot}}{a^3} = \Phi$$

All functions of B becomes now some constraints. Moreover

$$\Phi = \Phi_0(n) + B\Phi_1(n) + \ldots \quad \Rightarrow$$

$$\Phi = \Phi_0(n), \quad \frac{\partial \Phi}{\partial B} = \Phi_1(n)$$

$$\frac{\partial \Phi}{\partial n} = \frac{\partial \Phi_0}{\partial n}$$

Search for a nontrivial solution

Let

$$G^0 = 0 \quad \Rightarrow \quad \dot{R} = \ddot{R} = 0 \quad \Rightarrow \quad R = R_0$$

Then

$$\begin{cases} R_0 = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2+k}{a^2}\right) \\ T_0^0 + 3T_1^1 = 0 \quad \Rightarrow \quad \varepsilon - 3p - 3\lambda_1 n\frac{\partial\Phi_0}{\partial n} + 4\lambda_1\Phi_0 = 0 \\ T_0^0 = \frac{C}{a^4}, \qquad C = const \\ -12\gamma R_0\left(\frac{R_0}{12} + \frac{\dot{a}^2+k}{a^2}\right) = T_0^0 = \varepsilon + \lambda_1\Phi_0 \\ \left(\frac{\varphi'}{\varphi} - \frac{\varphi_1'}{\varphi_1}\right)(\varepsilon + p) + \lambda_1\frac{\varphi_1'}{\varphi_1} n\frac{\partial\Phi_0}{\partial n} + \lambda_1\Phi_1(n) = 0 \\ \varepsilon + p + \lambda_1 n\frac{\partial\Phi_0}{\partial n} + \varphi_1 n\dot{\lambda}_1 = 0 \\ \varphi_1\frac{(na^3)^{\cdot}}{a^3} = \Phi_0 \end{cases}$$

Search for a nontrivial solution II

$$R_0 = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right) \quad \Rightarrow$$

$$a^2 + k = -\frac{R_0}{12}a^2 + \frac{Q_0}{a^2} \quad \Rightarrow$$

$$C = -12\gamma R_0 Q_0$$

Trivial — $C = 0$ $\rightarrow$ Weyl vacuum ($T^\mu_\nu = 0$)

1. $Q_0 = 0 \quad \Rightarrow$
   de Sitter ($R_0 < 0$)
   AdS ($R_0 > 0$)
   Minkowski ($R_0 = 0$) $=$ Milne Universe
2. $Q_0 \neq 0 \quad \Rightarrow \quad R_0 = 0$

$a^2 + k = \frac{Q_0}{a^2} \quad \Rightarrow$

$$\pm(t - t_0) = \frac{1}{2}\int \frac{da^2}{\sqrt{Q_0 - ka^2}}$$

Search for a nontrivial solution III

$k = 0 \quad \Rightarrow \quad \pm(t - t_0) = \frac{a^2}{2\sqrt{Q_0}} \quad (Q_0 > 0)$

$k = +1 \quad \Rightarrow \quad \mp(t - t_0) = \sqrt{Q_0 - a^2} \quad (Q_0 > 0)$

$$a^2 + (t_0 - t)^2 = Q_0$$

$k = -1 \quad \Rightarrow \quad \pm(t - t_0) = \sqrt{Q_0 + a^2}$

$$(t - t_0)^2 - a^2 = Q_0$$

Non-trivial — $C \neq 0$

$$a^2 + k = -\frac{R_0}{12}a^2 + \frac{Q_0}{a^2}$$

$$\pm(t - t_0) = \frac{1}{2}\int \frac{da^2}{\sqrt{-\frac{R_0}{12}a^4 - ka^2 + Q_0}}$$

Non-trivial — $C \neq 0$

$$a^2 = -6\frac{k}{R_0} + \sqrt{-12\frac{Q_0}{R_0} - 36\frac{k^2}{R_0^2}}\,\sinh\left[\sqrt{-\frac{R_0}{3}}(t - t_0)\right]$$

if $R_0 < 0, \quad Q_0 > -3k^2/R_0$

$$a^2 = -6\frac{k}{R_0} + \sqrt{12\frac{Q_0}{R_0} + 36\frac{k^2}{R_0^2}}\,\cosh\left[\sqrt{-\frac{R_0}{3}}(t - t_0)\right]$$

if $R_0 < 0, \quad Q_0 < -3k^2/R_0$

$$a^2 = -6\frac{k}{R_0}\exp\left[\pm\sqrt{-\frac{R_0}{3}}(t - t_0)\right]$$

if $R_0 < 0, \quad Q_0 = -3k^2/R_0$

$$a^2 = -6\frac{k}{R_0} + \sqrt{-12\frac{Q_0}{R_0} + 36\frac{k^2}{R_0^2}}\,\sin\left[\sqrt{\frac{R_0}{3}}(t - t_0)\right]$$

if $R_0 > 0, \quad Q_0 < 3k^2/R_0$

Interpretation

Friedmann equations

$$\begin{cases} 3\frac{\dot{a}^2+k}{a^2} = 8\pi G T_0^0(eff) \\ 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2+k}{a^2} = 8\pi G T_1^1(eff) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} 8\pi G T_0^0(eff) = 3\frac{Q_0}{a^4} - \frac{R_0}{4} \\ 8\pi G T_1^1(eff) = -\frac{Q_0}{a^4} - \frac{R_0}{4} \end{cases}$$

The above makes sence only if some non-trivial solution exists

The existence

$$\begin{cases} a^2 + k = -\frac{R_0}{12}a^2 + \frac{Q_0}{a^2}, \quad C = -12\gamma R_0 Q_0 \\ \varepsilon - 3p + \lambda_1(4\Phi_0 - 3n\frac{\partial\Phi_0}{\partial n}) = 0 \\ \varepsilon + \lambda_1\Phi_0 = \frac{C}{a^4} \\ \varepsilon + p + \lambda_1 n\frac{\partial\Phi_0}{\partial n} + \varphi_1 n\dot{\lambda}_1 = 0 \ \Rightarrow \ \frac{4C}{a^4} + 3\varphi_1 n\dot{\lambda}_1 \\ \varphi_1\frac{(na^3)^{\cdot}}{a^3} = \Phi_0, \ \ \varepsilon + p = n\frac{\partial\varepsilon}{\partial n} \end{cases}$$

$$\varepsilon(n), \ n(t), \ \Phi_0(n), \ \lambda_1(t)$$

Let

$$\begin{cases} \varepsilon = 3p \ \Rightarrow \ \varepsilon = \varepsilon_0 n^{4/3}, \ \ (\lambda_1 \neq 0) \ \Rightarrow \\ \Phi_0 = \varphi_0 n^{4/3} \end{cases}$$

$$\begin{cases} \dot{\lambda}_1 = -\frac{4}{3}\frac{C}{\varphi_1}\frac{1}{a^4} \\ \\ \left((na^3)^{-1/3}\right)^{\cdot} = -\frac{1}{3}\frac{\varphi_0}{\varphi_1}\frac{1}{a} \end{cases}$$

Dark matter?

Dark energy?

The End

Thanks to Everybody