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# Gravitational Spin Hall Effect in Curves Spacetimes†

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**Abstract:** Geometric optics approximation sufficiently describes the effects in the near-earth environment, and Faraday rotation is purely a reference frame effect in this limit. A simple encoding procedure could mitigate the Faraday phase error. However, the framework of geometric optics is not sufficient to describe the propagation of waves of large but finite frequencies. So, we outline the technique to solve the equations for the propagation of an electromagnetic wave up to the subleading order geometric optics expansion in curved spacetimes. For this, we first need to construct a set of parallel propagated null tetrads in curved spacetimes. Then we should use the parallel propagated tetrad to solve the modified trajectory equation. The wavelength-dependent deviation of the electromagnetic waves is observed, which gives the mathematical description of the gravitational spin Hall effect.

**Keywords:** Geometric optics; spin Hall effect; Gravitational Faraday rotation; Spin optics; Parallel propagation; Fermi transport



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## 1. Introduction

Wigner rotation/phase, a special relativity effect, is the dominant source of relativistic errors during quantum communications in the near-Earth environment [1,2]. The typical magnitude of the Wigner phase during communication with the earth orbiting-Satellite (in some specific settings) was estimated to be of the order $10^{-4}$–$10^{-5}$ [3]. Another form of relativistic error is due to the gravitational polarization rotation (or gravitational Faraday effect) [4,5], which manifests in various astrophysical systems, like gravitational lensing phenomena [7] or accretion by astrophysical black holes [6]. The Gravitational Faraday effect has received numerous theoretical investigations, primarily within the geometric optics approximation [4,5,7–9], and also from the perspective of quantum communications[10]. Even if at the leading order, the gravitational Faraday rotation is pure gauge effect (depending on the emitter's and observer's orientation), one cannot simply disregard it [3].

Electromagnetic waves from astrophysical sources might have propagated through curved spacetimes. If the characteristic wavelength of the electromagnetic waves is small, but cannot be neglected, compared to the scale of the inhomogeneity of spacetime curvature, the necessity of the subleading order geometric optics correction arises. This subleading order correction includes wave effects, which affect both the propagation and polarization properties of waves [11]. Interaction of spin/polarization with external orbital angular momentum imparted by spacetime curvature results in the gravitational spin Hall effect [12–19]. Among various approaches to solving the wave equations in curved spacetimes and obtaining the gravitational spin Hall effect, we are particularly interested in generalizing the geometric optics that use WKB formalism. Ref. [15] first demonstrated this approach for stationary spacetimes and named the subleading order geometric optics correction as "spin optics", as it accounts for the spin-orbit coupling. Recently this formalism has been generalized to arbitrary spacetimes [20–22].

In Sec. 2, we present the WKB expansion for solving the electromagnetic wave equation in curved spacetimes. Then, we obtain the modified ray trajectory in the subleading order in Sec. 3. The polarization equation is obtained in Sec. 4 by generalizing the result that the solutions from the geometric optics approximation can be reduced to a set of Fermi propagated null tetrads to the subleading order. Finally, we discuss our results and conclude this article in Sec. 5.

We consider a spacetime manifold $M$ with the metric $g_{\mu\nu}$ of signature $(-,+,+,+)$ . The phase space in that manifold is the cotangent bundle $T^*M$, and its points are $(x,p)$. We write $\tilde{x}$ for the complex conjugate of $x$. We take $G=c=1$ and adopt the Einstein summation convention. A semicolon (;) denotes the covariant derivative along the curve, $\lambda$ denotes the parameter of electromagnetic wave curves and $\dot{x}=dx/d\lambda$. We use the sign convention for the curvature adopted in Ref. [23].

## 2. WKB formulation

We begin by writing the Maxwell equations in curved spacetime

$$F^{\alpha\beta}{}_{;\alpha} = -J^{\beta}, \tag{1}$$
$$F_{\alpha\beta;\gamma}+F_{\gamma\alpha;\beta}+F_{\beta\gamma;\alpha}=0. \tag{2}$$

Eq. (2) is identically satisfied if we write the electromagnetic field tensor $F^{\alpha\beta}$ in terms of the vector potential $A^{\alpha}$

$$F_{\alpha\beta}=A_{\beta;\alpha}-A_{\alpha;\beta}. \tag{3}$$

Now, let us use available gauge freedom, say the Lorenz gauge condition, in the Maxwell equations to constrain the vector potential $A^{\alpha}$, $A^{\alpha}{}_{;\alpha}=0$. Substituting this into Eq. (1), we obtain the equation for the electromagnetic wave

$$-A^{\alpha;\beta}{}_{;\beta}+R^{\alpha}_{\beta}A^{\beta}=J^{\alpha}, \tag{4}$$

where $R^{\alpha}{}_{\beta}$ is the Ricci tensor.

To solve the electromagnetic wave equation in the high-frequency regime, we start with the following ansatz for the vector potential

$$A^{\alpha}=a^{\alpha}e^{i\omega\mathcal{S}}, \tag{5}$$

where $a^{\alpha}$ is the complex amplitude that changes slowly, and $\omega\mathcal{S}$ is the real phase that varies rapidly. Here, $\omega$ represents the characteristic frequency of the wave. The gradient of the phase gives the wave vector, $l_{\alpha}=\mathcal{S}_{;\alpha}$. We write the polarization vector as $m^{\alpha}=a^{\alpha}/a$ and the square amplitude as $a=(\tilde{a}^{\alpha}a_{\alpha})^{1/2}$. After fixing notations for the wave vector and polarization vector, let us expand them in powers of $1/\omega$ as

$$l^{\alpha}=l^{\alpha}_{0}+\frac{l^{\alpha}_{1}}{\omega}+\frac{l^{\alpha}_{2}}{\omega^{2}}+..., \tag{6}$$
$$m^{\alpha}=m^{\alpha}_{0}+\frac{m^{\alpha}_{1}}{\omega}+\frac{m^{\alpha}_{2}}{\omega^{2}}+.... \tag{7}$$

First, we give the reason for separately expanding both the polarization and propagation vectors in powers of $\omega$. Unlike in conventional WKB expansion, higher order phase factors like $\mathcal{S}_1(\lambda)$ cannot be absorbed into the amplitude $m^{\alpha}_{0}$ by transformation $m^{\alpha}\rightarrow e^{i\mathcal{S}_1(\lambda)/\omega}m^{\alpha}$. The reason is that this transformation property will be constrained by the necessity to use the Fermi propagated null tetrad as follows

$$\frac{d\mathcal{S}_1(\lambda)}{d\lambda}=0. \tag{8}$$

Now, we substitute the vector potential from the WKB ansatz onto the Lorenz gauge condition, which gives

$$l^{\alpha}_{0}m_{0\alpha}+\frac{1}{\omega}\Big(l^{\alpha}_{0}m_{1\alpha}+l^{\alpha}_{1}m_{0\alpha}-i\Big(\frac{a_{;\alpha}}{a}m^{\alpha}_{0}+m^{\alpha}_{0;\alpha}\Big)\Big)=0, \tag{9}$$

up to the subleading order in $\omega$. Again, we substitute the vector potential into the source-free wave equation (Eq. (4) with $J^\alpha = 0$), which up to the subleading order in $\omega$, gives

$$m_0^\alpha l_{0\beta} l_0^\beta + \frac{1}{\omega}\left( m_1^\alpha l_{0\beta} l_0^\beta + 2 m_0^\alpha l_{1\beta} l_0^\beta - i\left( m_0^\alpha l_{0;\beta}^\beta + 2 m_{0;\beta}^\alpha l_0^\beta + 2\frac{a_{;\beta}}{a} m_0^\alpha l_0^\beta \right)\right) = 0. \tag{10}$$

We now calculate $\tilde{m}_{0\alpha} J^\alpha + m_{0\alpha} \tilde{J}^\alpha$, which is the identically vanishing quantity

$$l_{0\beta} l_0^\beta + \frac{2}{\omega}(l_{1\beta} - b_\beta) l_0^\beta = 0. \tag{11}$$

This equation could be seen as the generalized dispersion relation up to the subleading order geometric optics approximation. In obtaining this, we have substituted $\tilde{m}_0^\alpha m_{0\alpha} = 1$ and used

$$\frac{i}{2}\left(\tilde{m}^\alpha m_{\alpha;\beta} - m^\alpha \tilde{m}_{\alpha;\beta}\right) = i\tilde{m}^\alpha m_{\alpha;\beta} := b_\beta. \tag{12}$$

### 3. Propagation equation up to the subleading order

To obtain the subleading order correction to the trajectory equation, we start with the generalized dispersion Eq. (11). This is also the Hamilton-Jacobi equation for the phase $\mathcal{S}$ such that $\mathcal{S}_{;\alpha} = l_{0\alpha} + l_{1\alpha}/\omega$. The corresponding Hamiltonian on the phase space $T^*M$ is

$$H(x,l) = \frac{1}{2} g^{\alpha\beta} l_{0\alpha} l_{0\beta} + \frac{1}{\omega} g^{\alpha\beta}(l_{1\alpha} - b_\alpha) l_{0\beta} = \frac{1}{2\omega^2} g^{\alpha\beta}(\omega l_{0\alpha} + l_{1\alpha} - b_\alpha)(\omega l_{0\beta} + l_{1\beta} - b_\beta). \tag{13}$$

Hamilton's equations of motion are

$$\frac{dx^\alpha}{d\lambda} = \frac{\partial H}{\partial l_\alpha} = g^{\alpha\beta}\left( l_\beta - \frac{b_\beta}{\omega}\right), \tag{14}$$

and

$$\frac{dl_\alpha}{d\lambda} = -\frac{\partial H}{\partial x^\alpha} = \frac{1}{2}\dot{x}^\mu \dot{x}^\nu \frac{\partial g_{\mu\nu}}{\partial x^\alpha} + \frac{1}{\omega} g^{\mu\nu} \dot{x}_\nu \frac{\partial b_\mu}{\partial x^\alpha}, \tag{15}$$

where Eq. (14) is used to obtain the last equality. Hence, we could obtain the corresponding solution of the Hamilton-Jacobi equation (11) from [24]

$$\mathcal{S}(x,l) = \int_\lambda (\dot{x}^\alpha l_\alpha - H(x,l)) d\lambda = \frac{1}{2}\int \dot{x}^\alpha \dot{x}_\alpha d\lambda + \frac{1}{\omega}\int b_\alpha \dot{x}^\alpha d\lambda = \mathcal{S}_0 - \mathcal{S}_B, \tag{16}$$

where Eqs. (13) and (14) are used to obtain the third equality. Here, the additional phase $\mathcal{S}_B$ is polarization-dependent and could be viewed as the Berry phase, in analogy to the related phenomena in an inhomogeneous medium. The circularly polarized modes propagating in curved spacetimes acquire this geometric phase [25–27], which manifests dynamically, thereby giving the subleading order term in the propagation equation. This additional topological term results in noncollinear velocity and momentum (see Eq. (14)), a typical feature of waves propagating in anisotropic media (see, for example, [29]). Refs. [20], and [28] considered this form of action to obtain the spin Hall effect of electromagnetic waves. Also, in analogy with the spin-orbit coupling of light in gradient-index medium and that of electrons occurring in the Dirac equation (see, for example, Refs. [12,28,30,31]), we could identify $\mathscr{A}_\alpha = -b_\alpha = -i\tilde{m}^\beta m_{\beta;\alpha}$ with the Berry gauge field. The curvature associated with it

$$\frac{\partial \mathscr{A}_\alpha}{\partial x^\beta} - \frac{\partial \mathscr{A}_\beta}{\partial x^\alpha} = b_{\beta;\alpha} - b_{\alpha;\beta} := k_{\alpha\beta}. \tag{17}$$

could be identified with a field tensor associated with the vector potential $\mathscr{A}_\alpha$. Simplification of Eq. (15) gives [20]

$$\frac{D^2 x_\mu}{D\lambda^2} + \frac{1}{\omega}(b_{\mu;\nu} - b_{\nu;\mu})\dot{x}^\nu = 0 \implies \frac{D^2 x^\alpha}{D\lambda^2} \approx -\frac{i}{\omega} R^\alpha{}_{\beta\mu\nu} l_0^\beta m_0^\mu \tilde{m}_0^\nu, \tag{18}$$

where we have used

$$k_{\alpha\beta} := b_{\beta;\alpha} - b_{\alpha;\beta} = -iR_{\alpha\beta\mu\nu} m^\mu \tilde{m}^\nu + i\left(\tilde{m}^\nu{}_{;\alpha} m_{\nu;\beta} - \tilde{m}^\nu{}_{;\beta} m_{\nu;\alpha}\right), \tag{19}$$

to obtain the last equality. Thus, the electromagnetic wave propagates in a null but nongeodesic curve in the subleading order geometric optics.

## 4. Polarization equation up to the subleading order

Eqs. (A36) and (A37) are evolution equations for the propagation and polarization vectors, and to generalize them to the subleading order, let us write Fermi derivative operator $\mathcal{D}_{\dot{x}}$ along the ray $\dot{x}^\alpha$ [15]

$$\mathcal{D}_{\dot{x}} A^\alpha = l_0^\gamma A^\alpha{}_{;\gamma} - w_\gamma A^\gamma n^\alpha + A^\gamma n_\gamma w^\alpha, \tag{20}$$

where $A^\alpha$ is an arbitrary vector and $w^\alpha = l_0^\gamma \dot{x}^\alpha{}_{;\gamma}$ is an identically vanishing quantity in the geometric optics approximation. We have $\mathcal{D}_{\dot{x}} \dot{x}^\alpha = 0$ as $\dot{x}^\gamma \dot{x}_\gamma = 0$. A vector $A^\alpha$ is said to be Fermi propagated if its Fermi derivative $\mathcal{D}_{\dot{x}} A^\alpha$ is zero, and it can be shown that if two vectors are Fermi propagated, then their scalar product is constant. Let us consider this statement in the context of null tetrads $(\dot{x}^\alpha, n^\alpha, m^\alpha, \tilde{m}^\alpha)$: this set of null tetrads satisfies the orthogonality and completeness relations given in Eqs. (A33)-(A35) everywhere on the ray, provided that they are Fermi propagated, in which case they obey

$$l_0^\beta n^\alpha{}_{;\beta} = w^\beta n_\beta n^\alpha, \quad l_0^\beta m^\alpha{}_{;\beta} = w^\beta m_\beta n^\alpha. \tag{21}$$

Next, we fix $w^\beta n_\beta = 0$ by restricting the following transformation property of null tetrads

$$\dot{x}^\alpha \to F\dot{x}^\alpha, n^\alpha \to F^{-1} n^\alpha, \tag{22}$$

where $F$ is an arbitrary real function. This restriction in the transformation fixes the parameter $\lambda$ along the curve up to some rescaling $\lambda \to F^{-1}\lambda$. This way of choosing a parameter is known as canonical parametrization [20], which leads to the following relations

$$l_0^\beta n^\alpha{}_{;\beta} = 0, \qquad l_0^\beta m^\alpha{}_{;\beta} = w^\beta m_\beta n^\alpha. \tag{23}$$

These relations generalize Eqs. (A33)-(A37) of geometric optics.

The polarization vector is determined solely by the propagation direction or momentum of photons, while the momentum of particles is solely a function of position in curved spacetimes. Thus, the Berry connection determines how the polarization of a wave evolves in curved spacetimes as it is also the function of position only. We substitute $w^\alpha = l_0^\beta \dot{x}^\alpha{}_{;\beta}$ from Eq. (18) into Eqs. (23) to obtain the following equations for the evolution of the polarization vector

$$l_0^\beta n^\mu{}_{;\beta} = 0, \qquad l_0^\beta m^\mu{}_{;\beta} = \frac{i}{\omega} R_{\alpha\beta\gamma\delta} l_0^\alpha m_0^\beta m_0^\gamma \tilde{m}_0^\delta n_0^\mu. \tag{24}$$

From these equations, one can show that null tetrads $(\dot{x}^\alpha, n^\alpha, m^\alpha, \tilde{m}^\alpha)$ satisfies the normalization and orthogonality relations of Eqs. (A33)-(A35), up to the subleading order in $1/\omega$. However, the field does not satisfy all the polarization Eqs. (A43)-(A45), from which we can infer that it is not self-dual in the limit of spin optics [22].

In order to obtain a self-dual solution of the Maxwell equations (1) and (2) in the limit of spin optics, we define the Fermi-like derivative operator

$$\mathcal{D}'_{\dot{x}} A^\alpha = l_0^\beta A^\alpha_{;\beta} - w_\beta A^\beta n^\alpha + A^\beta n_\beta w^\alpha - \frac{i}{\omega}\left(\lambda_{;\mu} l^\mu m_\beta A^\beta m^\alpha - \tilde{\lambda}_{;\mu} l^\mu \tilde{m}_\beta A^\beta \tilde{m}^\alpha\right). \tag{25}$$

The vanishing of the Fermi-like derivative, $\mathcal{D}'_{\dot{x}} A^\alpha = 0$, of any two tetrad components $(\dot{x}^\alpha, n^\alpha, m^\alpha, \tilde{m}^\alpha)$ implies that the scalar product of these two components is preserved except that of $m^\alpha$ and $\tilde{m}^\alpha$ with their self. If tetrads with vanishing Fermi-like derivatives satisfy the following orthogonality and completeness relations at some point on the trajectory

$$\dot{x}^\alpha m_\alpha = \dot{x}^\alpha \dot{x}_\alpha = \dot{x}^\alpha \tilde{m}_\alpha = 0, \qquad m^\alpha \tilde{m}_\alpha = 1, \tag{26}$$
$$n^\alpha m_\alpha = n^\alpha n_\alpha = n^\alpha \tilde{m}_\alpha = 0, \qquad n^\alpha l_\alpha = -1, \tag{27}$$

then these relations hold throughout the trajectory. However, in general, the polarization vectors are not null anymore, $m^\alpha m_\alpha \neq 0$ and $\tilde{m}^\alpha \tilde{m}_\alpha \neq 0$, along the circularly polarized trajectory in the subleading order approximation. Moreover, these tetrad evolves as

$$l_0^\beta n^\alpha{}_{;\beta} = 0, \qquad l_0^\beta m^\alpha{}_{;\beta} = w^\beta m_\beta n^\alpha - \frac{i}{\omega} \tilde{\lambda}_{;\mu} l^\mu \tilde{m}^\alpha. \tag{28}$$

We could solve these equations to obtain [22]

$$m_1^\mu = (l_1^\alpha - b^\alpha) m_{0\alpha} n_0^\mu - i\tilde{\lambda} \tilde{m}_0^\mu, \qquad n_1^\mu = 0. \tag{29}$$

These components of tetrad describe the propagation of right-handed circularly polarized light rays in curved spacetime up to the subleading order geometric optics expansion. The reason is that they are the solution of the Maxwell equation in the Lorenz gauge and are self-dual (since they satisfy polarization equations (A43)-(A45)).

## 5. Discussion and conclusions

We have presented a WKB expansion procedure for extending the geometric optics approximation to include subleading order correction. This procedure involves the expansion of both the amplitude and phase in terms of the characteristic frequency $\omega$, which was necessary for the use of the Fermi-propagated tetrad. The requirement of parallel propagation of the null tetrad in the leading order generalizes to Fermi transport in the subleading order, constraining its transformation properties and resulting in the observer-independent spin Hall effect. Although the Hamiltonian equation (13) includes a gauge-dependent term, the propagation equation resulting from it is observer-independent. This is because, as explained in Eq. (8), we have restricted the $U(1)$ gauge freedom for the transformation of the polarization vector $m^\alpha$ to satisfy the requirement that it is Fermi propagated along the curve.

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## Appendix F Geometric optics approximation

We could substitute $m_1^\beta = 0 = l_1^\beta$ on equations. (9)-(11) to recover the complete equations of geometrical optics. The Lorenz condition (9) and the wave equation (10) in the leading order approximation becomes

$$l_0^\alpha m_{0\alpha} = 0 = l_0^\alpha l_{0\alpha}. \tag{A30}$$

Then, we calculate $\tilde{m}_{0\alpha} J^\alpha$ from Eq. (10) by taking $m_1^\alpha = 0 = l_{1\mu}$ as they are subleading order terms and could thus be neglected in geometric optics approximation, which gives

$$l_{0;\beta}^\beta + 2\tilde{m}_{0\alpha} m_{0;\beta}^\alpha l_0^\beta + 2\frac{a_{;\beta}}{a} l_0^\beta = 0. \tag{A31}$$

As $\tilde{m}_{0\alpha} m^{\alpha}_{0;\beta} l^{\beta}_{0}$ is purely imaginary and the remaining terms

$$l^{\beta}_{0;\beta} + 2\frac{a_{;\beta}}{a} l^{\beta}_{0},$$

are purely real, they should vanish separately vanish, from which we obtain

$$l^{\beta}_{0;\beta} + 2\frac{a_{;\beta}}{a} l^{\beta}_{0} = 0, \qquad m^{\alpha}_{0;\beta} l^{\beta}_{0} = 0. \tag{A32}$$

Following Eqs. (A30), we can construct a set of null tetrads $\left(l^{\alpha}_{0}, n^{\alpha}_{0}, m^{\alpha}_{0}, \tilde{m}^{\alpha}_{0}\right)$, where $l^{\alpha}_{0}$ and $m^{\alpha}_{0}$ could be identified with the propagation and polarization vectors, satisfying the following orthogonality and completeness relationships

$$l^{\alpha}_{0} m_{0\alpha} = l^{\alpha}_{0} l_{0\alpha} = l^{\alpha}_{0} \tilde{m}_{0\alpha} = 0, \qquad m^{\alpha}_{0} \tilde{m}_{0\alpha} = 1, \tag{A33}$$

$$m^{\alpha}_{0} m_{0\alpha} = \tilde{m}^{\alpha}_{0} \tilde{m}_{0\alpha} = 0, \tag{A34}$$

$$n^{\alpha}_{0} m_{0\alpha} = n^{\alpha}_{0} n_{0\alpha} = n^{\alpha}_{0} \tilde{m}_{0\alpha} = 0, \qquad n^{\alpha}_{0} l_{0\alpha} = -1. \tag{A35}$$

Auxiliary null vectors $n_{0\alpha}$ and $m_{0\alpha}$ are not unique and can be chosen to satisfy Eqs. (A34) and (A35). In Appendix G, we will show that circularly polarized waves satisfy equations (A34). Furthermore, the tetrad evolves as

$$l^{\alpha}_{0;\beta} l^{\beta}_{0} = 0, \qquad m^{\alpha}_{0;\beta} l^{\beta}_{0} = 0, \tag{A36}$$

$$n^{\alpha}_{0;\beta} l^{\beta}_{0} = 0, \tag{A37}$$

where Eqs. (A36) again results from geometric optics (we have used $l_{0\alpha;\beta} = l_{0\beta;\alpha}$ to obtain the first equation). $n^{\alpha}_{0}$ can be chosen such that equation (A37) is satisfied.

## Appendix G Self-dual and anti-self-dual fields

Let us construct the complex form of the field tensor $F^{\alpha\beta}$ as

$$\mathcal{F}^{s} = F + isF^{*}, \tag{A38}$$

where $F^{*}$ denotes the Hodge dual of $F^{\alpha\beta}$ and $s = \pm 1$. As the Hodge dual satisfies $(F^{*})^{*} = -F$, we can prove the relation $(\mathcal{F}^{s})^{*} = -is\mathcal{F}^{s}$. Any field satisfying this relation is called the self- (or anti-self-) dual antisymmetric field for $s = +1(\text{or} -1)$. One could expand a self-dual antisymmetric field in terms of the self-dual basis

$$(\mathbf{U}, \mathbf{V}, \mathbf{W}) = (\tilde{m} \wedge n, l \wedge m, m \wedge \tilde{m} - l \wedge n), \tag{A39}$$

as

$$\mathcal{F}^{+1} = \Phi_0 \mathbf{U} + \Phi_1 \mathbf{W} + \Phi_2 \mathbf{V}. \tag{A40}$$

In the geometric optics approximation, $\Phi_0 = \Phi_2 = 0$. Substituting the expression for the vector potential from Eq. (5) onto Eq. (A38), we obtain the following expression for the field $\mathcal{F}^{+1}_{\alpha\beta}$

$$\mathcal{F}^{+1}_{\alpha\beta} = i\omega \mathcal{Z}_{\alpha\beta} e^{i\mathcal{S}}, \tag{A41}$$

where

$$\mathcal{Z}_{\alpha\beta} = l_{\alpha} a_{\beta} - l_{\beta} a_{\alpha} - \frac{i}{\omega}\left(a_{\beta;\alpha} - a_{\alpha;\beta}\right). \tag{A42}$$

As the contraction of the self-dual field with the anti-self-dual field vanishes, we get

$$\mathcal{Z}_{\alpha\beta}m^{\alpha}n^{\beta} = 0, \tag{A43}$$

$$\mathcal{Z}_{\alpha\beta}\left(\tilde{m}^{\alpha}m^{\beta} - l^{\alpha}n^{\beta}\right) = 0, \tag{A44}$$

$$\mathcal{Z}_{\alpha\beta}l^{\alpha}\tilde{m}^{\beta} = 0. \tag{A45}$$

Complex conjugation of the amplitude $\mathcal{Z}_{\alpha\beta}$ of the self-dual $s = +1$ field gives an anti-self-dual $s = -1$ field. By substituting $\mathcal{Z}_{\alpha\beta}$ from Eq. (A42) to Eq. (A45), one can see that it is satisfied identically in geometric optics approximation. However, the substitutions onto equations (A43) and (A44) gives

$$m_0^{\alpha}m_{0\alpha} = 0 = l_0^{\alpha}m_{0\alpha}. \tag{A46}$$

These relations are presented in Eqs. (A33) and (A34) as the orthogonality conditions.

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