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Discretized Finsler Structure: An Approach to Quantize the First Fundamental Form †

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Abstract: Whether an algebraic or a geometric or a phenomenological prescription is applied, the first fundamental form is unambiguously related to the modeling of the curved spacetime. Accordingly, we assume that the possible quantization of the first fundamental form could be proposed. For precise accurate measure of the first fundamental form $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, the author derived a quantum-induced revision of the fundamental tensor. To this end, the four-dimensional Riemann manifold is extended to the eight-dimensional Finsler manifold, in which the quadratic restriction on the length measure is relaxed, especially in the relativistic regime and the minimum measurable length could be *ad hoc* imposed on the Finsler structure. The present script introduces an approach to quantize the fundamental tensor and first fundamental form. From gravitized quantum mechanics, the resulting relativistic generalized uncertainty principle (RGUP) is directly imposed on the Finsler structure, $F(\hat{x}_0^\mu, \hat{p}_0^\nu)$, which is obviously homogeneous of degree one in $\hat{p}_0^\mu$, the momentum of a test particle with mass $\bar{m} = m/m_{\text{p}}$ with m_{p} is the Planck mass. This unambiguously results in the quantized first fundamental form $d\tilde{s}^2 = [1 + (1 + 2\beta\hat{p}_0^\rho\hat{p}_{0\rho})\bar{m}^2(|\ddot{x}|/\mathcal{A})^2] g_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu$, where $\ddot{x}$ is the proper spacelike four-acceleration, $\mathcal{A}$ is the maximal proper acceleration, and β is the RGUP parameter. We conclude that an additional source of curvature associated with the mass $\bar{m}$, whose test particle is accelerated at $|\ddot{x}|$, apparently emerges. Thereby, quantizations of the fundamental tensor and first fundamental form are feasible.

Keywords: Modified theories of gravity; Noncommutative geometry; Curved spacetime; Relativity and gravitation

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1. Introduction

Following the assumption that the additional curvatures related to relativistic eight-dimensional spacetime tangent bundle $TM = M_4 \otimes M_4$ would be utilized to mimic the possible quantization on the four-dimensional spacetime M_4, the pseudo-Riemann manifold [1–4], the present script aims at introducing various possibilities to quantize the first fundamental form, the line element, of curved spacetime in the relativistic regime. To this end, we suggest the Finslerian manifold, which is a smooth n-dimensional differentiable manifold M_4 equipped with a continuous nonnegative Finsler norm $F: TM \rightarrow [0, +\infty)$ defined on the tangent bundle. For each point x on M_4, whose coordinates are $x^\mu = (ct, x^i)$, so that $(x^\mu, \dot{x}^\nu) \mapsto F(x^\mu, \dot{x}^\nu)$, where $\mu, \nu = 0, 1, 2, 3$ and $\dot{x^\nu} = \partial x^\nu / \partial s$ are tangent vectors and $\dot{x}^\mu \in T_x M$, with the tangent bundle $T_x M$ at x and $TM := U_{x \in M} T_x M$ is the tangent bundle on M, it is conjectured that $F(x^\mu, \dot{x}^\nu)$ satisfies three properties, namely

- positive definiteness, i.e., F is smooth on the complement of the zero section on TM,
- positive homogeneity, i.e., F is positively 1-homogeneous in $\dot{x}^\mu$, the relativistic four-velocity, i.e., $F(x^\mu, \lambda\,\dot{x}^\mu) = \lambda\, F(x^\mu, \dot{x}^\nu), \forall \lambda \in \mathbb{R}^+$, and
- subadditivity, i.e., for vectors $\vec{v}$ and $\vec{w}$ tangent to M_4 at the point x, F fulfills pointwise the triangle inequality $F(x^\mu, \vec{v} + \vec{w}) \leq F(x^\mu, \vec{v}) + F(x^\mu, \vec{w})$.

The Hessian of $F^2(x^\alpha, \dot{x}^\beta)$ determines the Finsler metric,

$$g_{AB} = \frac{1}{2}\frac{\partial^2}{\partial \dot{x}^\alpha \partial \dot{x}^\beta} F^2(x^\alpha, \dot{x}^\beta), \tag{1}$$

with $A, B = 0, 1, 2, \cdots, 7$, while $\alpha, \beta = 0, 1, 2, 3$ and the resulting g_{AB} is positive.

Our approach to derive quantum-induced revisiting metric tensor shall be outlined in section 2. For now, some comments on the Finsler manifold are now in order.

- First, the Finsler manifolds characterized by M_4 and $F^2(x^\mu, \dot{x}^\nu)$. With this we mean that the Finsler manifold is composed of i) a base space and ii) a real scalar-valued function F. The base space is a set of positions in $\mathbb{R}^4$. The real scalar-valued function $\mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^+$ captures the additional structure of the space.
- Second, on TM, the covariant derivatives represent the standard operators of the Heisenberg algebra and the components of the curvature tensor represent the noncommutation relations [5–7].
- Third, the Finsler geometry, which is Riemann geometry but with relaxed quadratic measure restriction, is also concerned with measuring distances on abstract spaces. In the context of the present script, we recall that the distance between two points on Finsler manifold is defined in a similar manner to the standard Euclidean distance, i.e., the length of the shortest curve connecting those two points. On Euclidean manifold, the length of a curve is a sum over infinitesimal line elements ds. On Finsler manifold, on the other hand, the summation over ds is weighted depending on position and direction. Therefore, the Finsler geometry is formulated with the directions of the tangent vectors $\dot{x}^\mu$ but not with their magnitudes. This leads to two kinds of affine connections. One is with respect to the infinitesimal changes of the directional variables. The other one is with respect to the infinitesimal changes of the coordinates.

A possible discretization of the spacetime manifold is based on qravitized quantum mechanics, i.e. relativistic generalized uncertainty principle (RGUP) [8–11]. In section 2, we first introduce our approach to *continuous* manifolds. This is an almost *ad hoc* imposing of the minimum measurable length on the *continuous* Finsler structure.

2. First Fundamental form on Continuous Finsler Manifold

The proposed approach is based on the existence of a minimum measurable length, which was proved in loop quantum gravity, doubly-special relativity, and string theory, for instance. The minimum measurable length could be interpreted as a nonvanishing position uncertainty that emerged from the impacts of finite gravitational fields on the Heisenberg uncertainty principle, the fundamental theory of quantum mechanics [8–11].

The Finsler structure of the Riemann manifold is conjectured to satisfy

$$F^2(x, \dot{x}) \;=\; g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu, \tag{2}$$

where $g_{\mu\nu}(x)$ is Finsler metric which is apparently distinct from the Riemann metric $g_{\mu\nu}$. Then, the length of the curve $s : [0, L] \to M_4$ is given as

$$\mathcal{L}(s) \;=\; \int_0^L F\left(s(t), \frac{\dot{s}(t)}{||\dot{s}(t)||}\right) ||\dot{s}(t)|| dt, \tag{3}$$

where $\dot{s}(t) = ds(t)/dt$, $t \mapsto s(t)$ on M_4, and the tangent norm does not need to be induced by inner product. The Euclidean length of that curve can be deduced from Eq. (3), at $F = 1$.

As F at any point $(x, \dot{x})$ is positively homogeneous of degree one in $\dot{x}(t)$,

$$\mathcal{L}(x) \;=\; \int_0^L F(x(t), \dot{x}(t)) dt, \tag{4}$$

so that the function F is locally acting on the first fundamental form ds. The resulting length of the curve does not depend on the choice of the parameter t along the curve measure.

The Riemannian geometry is obtained as a particular case, namely, $F^2(x^\mu, \dot{x}^\nu) = g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu$. Riemann metric $g_{\mu\nu}$ and the Finsler metric $g_{\mu\nu}(x)$, are equal, especially at the point x. This is not the case with general Finsler metrics. Both $F(x^\mu, \dot{x}^\nu)$ and $g_{\mu\nu}(x)$ determine the Finslerian geometry, while the Riemannian geometry is merely derived from $g_{\mu\nu}$.

With the finite positive minimum measurable length

$$\Delta x_{\mathtt{min}} = \sqrt{-|g||\beta_0|}\ell_{\mathtt{p}}, \tag{5}$$

where $\ell_{\mathtt{p}}$ is the Planck length, g is the fundamental metric and $|\beta_0|$ is a dimensionless RGUP parameter that can be determined from cosmological observations [11,12] or table-top laboratory experiments [13], the Finsler structure reads

$$F(x^\mu, \sqrt{-|g||\beta_0|}\ell_{\mathtt{p}}\, \dot{x}^\mu) = \sqrt{-|g||\beta_0|}\ell_{\mathtt{p}}\, F(x^\mu, \dot{x}^\mu), \qquad \forall \sqrt{-|g||\beta_0|}\ell_{\mathtt{p}} \in \mathbb{R}^+. \tag{6}$$

Other RGUP approaches have been discussed in literature [14–16].

As for the RGUP approach proposed by the author [17–19],

- $\sqrt{-|g|}$ characterize the relevance to curved spacetime,
- assigns physical dimensions to the spacetime coordinates of the general theory of relativity, and
- assures physical interpretation independent on the choices of coordinates, $|\beta_0|$ introduces the consequences of gravity to the relativistic Heisenberg uncertainty principle [17–19].

Accordingly, the local coordinates on TM are expressed as

$$x^A = \left(x^\alpha, \sqrt{-|g||\beta_0|}\, \dot{x}^\alpha\right), \tag{7}$$

and the first fundamental form, the infinitesimal distance in the relativistic eight-dimensional spacetime tangent bundle TM, reads,

$$d\tilde{s}^2 = g_{AB}\, dx^A\, dx^B. \tag{8}$$

Equation (7) introduces a relation between eight- to four-dimensional spaces.

With the eight parametric equations

$$x^A = x^A(\zeta), \tag{9}$$

$$\dot{x}^\alpha = \frac{\partial x^\alpha(\zeta)}{\partial \zeta^\mu}\dot{\zeta}^\mu, \tag{10}$$

the four-coordinates x^μ on TM are correlated with the parameterization ζ, four-coordinates parameterizing the four-dimensional spacetime manifold M_4 [5], the counterpart first fundamental form, line element, can be parameterized as

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu}\, d\zeta^\mu\, d\zeta^\nu. \tag{11}$$

Then, by equating (8) and (11), we get

$$\tilde{g}_{\mu\nu} = g_{AB}\, \frac{\partial x^A(\zeta)}{\partial \zeta^\mu}\, \frac{\partial x^B(\zeta)}{\partial \zeta^\nu}. \tag{12}$$

Differentiating the eight–dimensional coordinates on TM with respect to the four-dimensional coordinates ζ^{μ} on M_4 determines the quantum-induced corrections to the fundamental tensor

$$\tilde{g}_{\mu\nu} = \left[1 + \left(-|g\beta_0|\ell_{\mathrm{p}}^2\right)|\ddot{x}|^2\right] g_{\mu\nu}. \tag{13}$$

where $|\ddot{x}|^2 \equiv \ddot{x}^{\lambda}\ddot{x}_{\lambda} = g_{\delta\gamma}\ddot{x}^{\delta}\ddot{x}^{\gamma}$ with λ, δ, γ are dummy indices and $\ddot{x}^{\lambda} = \partial\dot{x}^{\lambda}/\partial\zeta^{\lambda}$. $\ddot{x}^{\lambda}$ could be interpreted as proper spacelike four-acceleration [20,21]. Alternatively, $\ddot{x}^{\lambda}$ would be interpreted as geodesic related to the additional curvature. To avoid any controversial discussion about the physical meaning of $\ddot{x}^{\lambda}$, we suggest normalizing $\ddot{x}^{\lambda}$ to $-|g\beta_0|\ell_{\mathrm{p}}^2$. A second reason for this normalization would be the conservative representation of the key results concluded in the present script that we are presenting quantum-induced corrections but not a full quantization.

To summarize the present section, we conclude that the first fundamental form reads

$$d\tilde{s}^2 = \left[1 + \left(-|g\beta_0|\ell_{\mathrm{p}}^2\right)|\ddot{x}|^2\right] g_{\mu\nu} dx^{\mu} dx^{\nu}. \tag{14}$$

As discussed, Eq. (13) refers to quantum-induced revisiting metric tensor derived deduced from *continuous* Finsler manifold.

Section 3 introduces an improvement based on proper inclusion of RGUP, not just its minimum measurable length, within the curved spacetime, profound contributions of the present study.

3. First Fundamental Form on Discretized Finsler Manifold

In section 2, we have introduced our approach based on the emergence of an additional curvature on the relativistic eight-dimensional spacetime tangent bundle TM, which is equipped with a continuous nonnegative Finsler norm $F : TM \to [0, +\infty)$. Also, we have introduced the assumption that both infinitesimal distances on TM and M_4 are identical so that

$$d\tilde{s}^2 = g_{AB}\, dx^{A}(\zeta)\, dx^{B}(\zeta) = \tilde{g}_{\mu\nu}\, d\zeta^{\mu}\, d\zeta^{\nu}. \tag{15}$$

We realized the measure of $d\tilde{s}^2$ is apparently precise without quantum nature.

In order to endow quantum nature to this measure, we are urgent to

- express the metric tensor as an operator, especially that the 1-form could be an operator,
- suggest noncommutative relations for these quantities, and
- integrate probability distributions and quantum superposition.

In this regard, we emphasize that neither the metric tensor nor the 1-form, dx^{μ}, has noncommutative translations [22]. Alternatively, we might recall noncommutative metric tensor [23] and noncommutative differential calculus [24,25] to define a noncommutative line element [26]. This could be done elsewhere.

To remain within the scope of the present script, we resume with the RGUP approach, which was introduced in section 2. Here, we concretely aim at discretizing the eight-dimensional tangent bundle. For a test particle with mass m normalized to the Planck mass $\bar{m} = m/m_{\mathrm{p}}$, the Finsler structure reads

$$F\left(\hat{x}^{\mu}, \bar{m}\hat{\dot{x}}^{\nu}\right) = F(\hat{x}^{\mu}, \hat{p}^{\nu}), \qquad \forall \bar{m} \in \mathbb{R}^{+}, \tag{16}$$

where $\hat{x}^{\mu} = \mathbf{x}^{\mu} = \hat{x}_0^{\mu} = (\hat{x}_0^0, \hat{x}_0^i) = (ct, \hat{x}_0^i)$ and $\hat{p}^{\nu} = -i\hbar\partial/\partial\hat{x}^{\nu} = \hat{p}_0^{\nu} = (\hat{p}_0^0, \hat{p}_0^i) = (E/c, \hat{p}_0^i)$. These are the typical definitions in ungravitized QM and most probably remain unchanged in quantized QM.

On Finsler manifold with RGUP and the parameterization $x^A = x^A(\zeta)$, Eq. (1) reads

$$g_{AB} = \frac{1}{2}\frac{\partial^2}{\partial \hat{p}_0^\mu \partial \hat{p}_0^\nu} F^2(\hat{x}_0^\mu, \hat{p}_0^\nu(1+\beta \hat{p}_0^\rho \hat{p}_{0\rho})) \tag{17}$$

$$= \left(1+\beta \hat{p}_0^\rho \hat{p}_{0\rho}\right) \frac{1}{2}\frac{\partial^2}{\partial \hat{p}_0^\mu \partial \hat{p}_0^\nu} F^2\left(\hat{x}_0^\mu, \hat{p}_0^\nu\right), \tag{18}$$

where $\beta = \beta_0 G/(c^3\hbar) = \beta_0(\ell_{\text{p}}/\hbar)^2$ is the RGUP parameter, G is the gravitational constant, c is the speed of light, $\hbar$ is the Planck constant, and ℓ_{p} is the Planck length [8–11]. Eq. (18) assumes that $F(\hat{x}_0^\mu, \hat{p}_0^\nu)$ is homogeneous of degree one in $\hat{p}_0^\mu$ and $\left(1+\beta \hat{p}_0^\rho \hat{p}_{0\rho}\right) > 0$. Both $\hat{x}_0^\alpha$ and $\hat{p}_0^\beta$ are also parameterized with ζ.

If we limit the discussion on the Finsler structure of Riemann manifold, Eq. (2), [27], we get

$$F(\hat{x}_0^\mu, \hat{p}_0^\nu) = \left[\frac{|\hat{p}_0^\nu|^2 - |\hat{x}_0^\mu|^2|\hat{p}_0^\nu|^2 + \left(\hat{x}_0^\mu \cdot \hat{p}_0^\nu\right)^2}{1-|\hat{x}_0^\mu|^2}\right]^{1/2}, \tag{19}$$

where RGUP suggests that $\hat{p}_0^\nu = \hat{p}_0^\nu(1+\beta \hat{p}_0^\rho \hat{p}_{0\rho}))$ [8,9]. Then Eqs. (12) and (13) lead to

$$\tilde{g}_{\mu\nu} = \frac{1}{2}\frac{\partial^2}{\partial \hat{p}_0^\mu \partial \hat{p}_0^\nu} \left[\frac{\sum_{\nu=0}^3(\hat{p}_0^\nu)^2 - \sum_{\mu=0}^3(\hat{x}_0^\mu)^2 \sum_{\nu=0}^3(\hat{p}_0^\nu)^2 + \left(\sum_{\mu|\nu=0}^3 \hat{x}_0^\mu \hat{p}_0^\nu\right)^2}{1-\sum_{\mu=0}^3(\hat{x}_0^\mu)^2}\right]$$
$$\left[\frac{d\hat{x}_0^\mu(\zeta^\mu)}{d\zeta^\mu}\frac{d\hat{x}_0^\nu(\zeta^\nu)}{d\zeta^\nu} + \left(1+2\beta \hat{p}_0^\rho \hat{p}_{0\rho}\right)\bar{m}^2 \frac{d\dot{\hat{x}}_0^\mu(\zeta^\mu)}{d\zeta^\mu}\frac{d\dot{\hat{x}}_0^\nu(\zeta^\nu)}{d\zeta^\nu}\right]. \tag{20}$$

The quantized metric tensor, Eq. (20), could be approximated as

$$\tilde{g}_{\mu\nu} \simeq \left[1+\left(1+2\beta \hat{p}_0^\rho \hat{p}_{0\rho}\right)\bar{m}^2\left(\frac{|\ddot{x}|}{\mathcal{A}}\right)^2\right] g_{\mu\nu}, \tag{21}$$

where $\mathcal{A}$ is the maximal proper acceleration [20,21,28–30].

Relative to Eq. (13), Eq. (20) refers to a full-quantized version of the fundamental tensor, which is obtained when RGUP is properly imposed on Finster structure. Then, the first fundamental form reads

$$d\tilde{s}^2 = \left[1+\left(1+2\beta \hat{p}_0^\rho \hat{p}_{0\rho}\right)\bar{m}^2\left(\frac{|\ddot{x}|}{\mathcal{A}}\right)^2\right] g_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu. \tag{22}$$

We conclude that finite $|\ddot{x}|$ and $\bar{m}$ are essential for the quantization of the fundamental tensor, $\tilde{g}_{\mu\nu}$, Eq. (21), and first fundamental form, $d\tilde{s}^2$, Eq. (22). In the relativistic regime, in which the approach of RGUP and hence the spacetime quantization are possible, an additional source of spacetime curvature apparently emerges. This is the curvature associated with the mass $\bar{m}$, whose test particle's motion has acceleration $|\ddot{x}|$. Vanishing $|\ddot{x}|$ and $\bar{m}$ entirely restore the unquantized versions of the fundamental tensor, $g_{\mu\nu}$, and first fundamental form, ds^2.

As for the relativistic formulation of GUP [17–19], comments on ref. [31] are now in order. This script studies the inconsistency of the original HUP approach with special relativity and characterizes the motion of a particle crossing the worldline. The resulting minimal measurable length is suggested as $1/2mc^2$. Comparing to Eq. (5) this estimation seems not manifesting various relativity principles, for the example, that the coordinates in GR are fundamentally arbitrary.

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declaration

The author declares

- no conflicts of interest regarding the publication of this paper!
- compliance with ethical standards regarding the contents of the present script!
- no data is associated with the present script.

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