GLOBAL WELL-POSEDNESS FOR 3D-AXISYMMETRIC ANISOTROPIC BOUSSINESQ SYSTEM WITH STRATIFICATION EFFECTS

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Oussama Melkemi

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Boussinesq System

The $3d-$Boussinesq system :

$$\left\{\begin{array}{ll}\partial_t v + v\cdot\nabla v - \upsilon\Delta v + \nabla p = \rho\vec{e_3} & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^3,\\ \partial_t \rho + v\cdot\nabla\rho - \kappa\Delta\rho = -\mathcal{N}^2(x_3)v_3 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^3,\\ \operatorname{div} v = 0, & \\ (v,\rho)_{|t=0} = (v_0,\rho_0). & \end{array}\right. \tag{B}$$

- $v = (v_1, v_2, v_3)$ describes the velocity
- $v\cdot\nabla \triangleq \sum_{i=1}^{3} v_i\partial_i$
- ρ represents the density of the fluid
- p is the pressure
- υ, κ denotes viscosity and thermal diffusivity
- $\mathcal{N}^2$ is a scalar function called the stratification frequency
- $\operatorname{div} v = 0$ means that the fluid is incompressible

Navier-Stokes equations

The Navier-Stokes equations :

$$\begin{cases} \partial_t v + v\cdot\nabla v - \upsilon\Delta v + \nabla p = 0 \quad \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^d, \\ \mathrm{div} v = 0, \\ v_{|t=0} = v_0. \end{cases} \tag{NS}$$

1. **The Beale-Kato-Majda Criterion :**
   - Let $w = rot(v)$ and $v_0 \in H^s(\mathbb{R}^d), s > \frac{d}{2}+1$.

     The lifespan T_* is finite $\Rightarrow \int_0^{T_*} \|w(\tau)\|_{L^\infty} d\tau = +\infty$,

2. **The vorticity formulation :**
   - In the 2-dimensional case : $w = \partial_1 v_2 - \partial_2 v_1$:

     $$\partial_t w + v\cdot\nabla w - \upsilon\Delta w = 0$$

   - In the 3-dimensional case :

     $$\partial_t w + v\cdot\nabla w - \upsilon\Delta w = \underbrace{-w\cdot\nabla v}_{\textbf{Streatching term}}.$$

Axisymmetric data without swirl

- We say that the velocity is axisymetric without swirl if : $v(t,x_1,x_2,x_3) \triangleq v_r(t,r,z)\vec{e}_r + v_z(t,r,z)\vec{e}_z$,
- with $x_1 = r\cos\theta, \quad x_2 = r\sin\theta$ and $x_3 = z$,
- $r = \sqrt{x_1^2 + x_2^2}, \quad 0 \leq \theta \leq 2\pi$.
- The triplet (e_r, e_θ, z) is the cylindrical basis given by $\vec{e}_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0), \quad \vec{e}_\theta = (-\sin\theta, \cos\theta, 0)$ and $\vec{e}_z = (0,0,1)$.
- $w \triangleq (\partial_z v_r - \partial_r v_z)e_\theta = w_\theta e_\theta$.
- $v \cdot \nabla \triangleq v_r\partial_r + v_z\partial_z$
- $\mathrm{div}(v) = \partial_r v_r + \frac{v_r}{r} + \partial_z v_z$
- Stretching term becomes :

$$w \cdot \nabla v = \frac{v_r}{r} w.$$

Interesting results

1. **Navier-Stokes :**
   1. $(\upsilon = 0)$: J.Leray : Weak-Solutions.
   2. Axisymmetric data :
      - $(\upsilon = 0)$: T.Shirota and T.Yanagisawa : Global existence
      - $(\upsilon > 0)$: O. Ladyzhenskaya : Unique solvability.
2. **Axisymmetric-Boussinesq system :**
   - $(\upsilon > 0, \kappa = 0, \mathcal{N} = 0)$: H. Abidi, T. Hmidi and S. Keraani : Global regularity.
   - $(\upsilon > 0, \kappa \geq 0, \mathcal{N} = 0)$: T. Hmidi and F. Rousset : Global existence.
   - $(\upsilon = 0, \kappa > 0, \mathcal{N} = 0)$: T. Hmidi and F. Rousset : Global well-posedness.

Anisotropic Boussinesq system

- The $3d-$Anisotropic Boussinesq system :

$$\begin{cases} \partial_t v + v\cdot\nabla v - \upsilon_h\Delta_h v + \nabla v = \rho\vec{e_3} & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^3, \\ \partial_t\rho + v\cdot\nabla\rho - \kappa_h\Delta_h\rho = -\mathcal{N}^2(x_3)v_3 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^3, \\ \operatorname{div} v = 0, \\ (v,\rho)_{|t=0} = (v_0,\rho_0). \end{cases} \qquad (B_{\upsilon_h,\kappa_h})$$

- $\Delta_h \triangleq \partial_{x_1}^2 + \partial_{x_2}^2$.
- In the cylindrical coordinates : $\Delta_h \triangleq \partial_{rr} + \frac{\partial_r}{r}$.
- $(\mathcal{N}=0)$: Miao and Zheng : Global well posedness for $(B_{\upsilon_h,\kappa_h})$.

Main result and idea of proof

1 Main Result :

- $v_0 \in H^1$ be an axisymmetric divergence free vector field without swirl
- $\frac{w_0}{r} \in L^2, \partial_z w_0 \in L^2$
- $\rho_0 \in H^{0,1}$ an axisymmetric function.
- Then the system $(B_{\upsilon_h,\kappa_h})$ admits a unique global solution such that $(v,\rho) \in \big(\mathcal{C}(\mathbb{R}_+;H^1) \cap L^2_{loc}(\mathbb{R}_+;H^{1,2} \cap H^{2,1}) \cap L^1_{loc}(\mathbb{R}_+;Lip)\big) \times \mathcal{C}(\mathbb{R}_+;H^{0,1} \cap H^{1,1})$.

2 Idea of proof

- The vorticity formulation :

$$\partial_t w_\theta + v\cdot\nabla w_\theta - \Delta_h w_\theta + \frac{w_\theta}{r} = -\partial_r \rho + \frac{v_r}{r} w_\theta.$$

- In particular,

$$\partial_t \xi + v\cdot\nabla \xi + \mathcal{A}_{h,r}\xi = -\frac{\partial_r}{r}\rho,$$

- $\xi \triangleq \frac{w_\theta}{r}$ and $\mathcal{A}_{h,r} = -(\Delta_h + \frac{2\partial_r}{r})$.

* Coupling the system $(B_{\upsilon_h,\kappa_h})$:

$$\partial_t\widetilde{\Gamma}+v\cdot\nabla\widetilde{\Gamma}-\Delta_h\widetilde{\Gamma}+\frac{2}{r}\partial_r\widetilde{\Gamma}=-\frac{1}{2}v_z,\quad \widetilde{\Gamma}\triangleq\zeta-\frac{\rho}{2}.$$

1. **A priori estimate :**

   1.2 We need to control the **Lipschitz norm** of the velocity :

   - L^2-estimates for velocity and density
   - Vertical information for density and vorticity.
   - By using the coupled function : L^2-estimate of $\xi=\frac{w_\theta}{r}$.

2. **Existence of solutions** :
   - Using the Friedrichs method.
   - Estimates are uniformly controlled with respect to the parameter n.
   - Cauchy-Lipschitz Theorem.
   - By standard compactness arguments, we obtain the converges.

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THANK YOU FOR YOUR ATTENTION.