Existence and attractivity results for fractional differential inclusions via nocompactness measures

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Objective

In this work, we consider the following differential inclusion with initial condition

$$(1) \qquad \begin{cases} \mathcal{D}^{\gamma}\left(\frac{dw}{d\phi}\right)(z) \in G(z, w(z)); \quad z \in (1, +\infty) \\ \frac{dw}{d\phi}(1) = w_0; \quad w(1) = w_1 \end{cases}$$

where $w_0, w_1 \in \mathbb{R}$, $\frac{dw}{d\phi}$ is the Stieltjes derivative of w with respect to ϕ, $\mathcal{D}^{\gamma}$ is the Hadamard fractional derivative of order $0 < \gamma < 1$, $G : [1, +\infty) \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is multivalued map and $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

Objective

We will study the existence and the stability of the solutions for the differential inclusion in an appropriate functional space. Our main tools are nocompactness measures combined with fixed point theorems.

Study steps

1. Define a functional space and choose an appropriate nocompactness measure.
2. Consider sufficient conditions to obtain the existence of solutions to our problem.
3. Reduce the search for the existence of these solutions to the search for the existence of the fixed points of operators defined on this functional space.
4. The choice of the nocompactness measure allows us to characterize these solutions in a certain sense of stability.

Plan de présentation

1. Generalities
2. Main results
3. Exemple

Generalities

Assume that $\phi : \mathbb{R} \to \mathbb{R}$ is monotone, nondecreasing and continuous from the left everywhere. D_ϕ is the set of discontinuity points of ϕ.

Definition

A function $u : \Omega \subset \mathbb{R} \to \mathbb{R}$ is ϕ-continuous at a point $z_0 \in \Omega$ (or continuous with respect to ϕ at z_0) if for every $\epsilon > 0$, there exists $\rho > 0$ such that

$$(2) \qquad z \in \Omega; \quad |\phi(z) - \phi(z_0)| < \rho \Rightarrow |u(z) - u(z_0)| < \epsilon.$$

We say that u is ϕ-continuous on Ω if it is ϕ-continuous at every point $z_0 \in \Omega$.

Generalities

We denote by $\mathcal{B}(J)$, the Banach space of bounded functions on the interval $J = [1, +\infty)$ equipped with the norm of uniform convergence, and by $\mathcal{B}_\phi(J)$ the subspace of bounded functions which are also ϕ-continuous on J.

Theorem

$\mathcal{B}_\phi(J)$ is a Banach space.

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Theorem

$\mathcal{B}_\phi(J)$ is a Banach space.

Proof.

It suffices to show that $\mathcal{B}_\phi(J)$ is a closed of the Banach space $\mathcal{B}(J)$. $\square$

Generalities

Definition

A function $u : J \to \mathbb{R}$, is said to be a regulated function if for every $z \in J^+$ the right-sided limit $u(z^+) := \lim\limits_{t \to z^+} u(t)$ exists and for every $z \in J^-$ the left-sided limit $u(z^-) := \lim\limits_{t \to z^-} u(t)$ exists.

Denote by $\mathcal{RB}(J)$ the Banach space consisting of all bounded and regulated real functions defined on J with the norm of uniform convergence.

Generalities

Theorem

A nonempty subset $\Omega \subset \mathcal{RB}(J)$ is relatively compact if and only if the following three conditions are satisfied :

(1) *The set $\Omega(z)$ is bounded.*

(2) *For each $T > 1$ the set $\Omega|_{[1,T]}$ is equiregulated.*

(3) *For each $\epsilon > 0$ there are $\rho > 0$ and $T > 1$ such that, for any $u, v \in \Omega$, if $\sup_{z\in[1,T]} |u(z) - v(z)| < \rho$ then $\|u - v\| < \epsilon$.*

Generalities

Definition

If $G : X \to \mathcal{P}_{cl}(Y)$ is a multivalued operator, a selection (or selector) of G is a singlevalued operator $\varphi : X \to Y$ such that $\varphi(z) \in G(z)$, for each $z \in X$.

In many fields of both the theory and applications of multifunctions it is extremely important to ensure the existence of selections with special additional properties.

Theorem

Let $G : X \to \mathcal{P}_{cl,cv}(Y)$ be a lower semicontinuous multifunction. Then G admits continuous selection.

Generalities

Theorem

Let Ω be a nonempty, bounded, closed and convex subset of the Banach space X and let $G : \Omega \longrightarrow \mathcal{P}_{cl,cv}(\Omega)$ be a closed. Assume that there exists a constant $k \in [0,1)$ such that $\psi(GA) \leq k\psi(A)$ for any nonempty subset A of Ω. Then G has a fixed point in the set Ω.

Remarks

Let us denote by Fix G the set of all fixed points of the operator G which belong to Ω. The set Fix G belongs to the family ker ψ.

Main results

consider the function ψ defined on the family $\mathcal{M}_{\mathcal{B}_\phi(J)}$ by the formula

$$(3) \qquad \psi(\Omega) = \max\left\{\omega_\phi(\Omega), \omega_\phi^+(\Omega)\right\} + \limsup_{z\to\infty} diam\, \Omega(z).$$

Theorem

The mappings ψ is a measure of nocompactness in the space $\mathcal{B}_\phi(J)$.

Proof.

Take $\Omega \in \mathcal{M}_{\mathcal{B}_\phi(J)}$. We show that if $\psi(\Omega) = 0$, Ω satisfies the conditions in Theorem 4. Since ϕ is left continuous, $\max\left\{\omega_\phi(\Omega), \omega_\phi^+(\Omega)\right\} = 0$ ensure that Ω is equiregulated and $\limsup_{z\to\infty} diam\, \Omega(z) = 0$ implies (3). □

Main results

Consider the following inclusion

$$u(z) \in (Fu)(z);\ z \in J. \tag{4}$$

Definition

The solution $u = u(z)$ of (4) is said to be globally uniformly attractive if for each solution $v = v(z)$ of (4) we have that for each $\epsilon > 0$ there exists $T > 1$ such that

$$\big|u(z) - v(z)\big| < \epsilon,\ z \geq T. \tag{5}$$

Main results

Remarks

The kernel ker ψ *consists of nonempty and bounded sets* Ω *such that functions from* Ω *are locally equiregulated on* J *and for each* $u, v \in \Omega$, *(5) hold.*

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The kernel ker ψ consists of nonempty and bounded sets Ω such that functions from Ω are locally equiregulated on J and for each $u, v \in \Omega$, (5) hold.

Let $\frac{dw}{d\phi}(z) = v(z)$ then $w(z) = w_0 + \int_1^z v(t)d\phi(t)$ and the differential inclusion (1) can be written as

$$(6) \qquad v(z) \in w_1 + \frac{1}{\Gamma(r)} \int_1^z \left(\ln \frac{z}{t}\right)^{\gamma-1} G\left(t, w_0 + \int_1^t v(\tau)d\phi(\tau)\right) \frac{dt}{t},$$

Main results

The differential inclusion (1) will be considered under the following assumptions :

(H_1) There exist continuous and bounded functions $p, q : J \to \mathbb{R}_+$ such that

$$H(G(z,u), G(z,v)) \leq p(z)|u - v|; \; z \in J,$$

for all $u, v \in \mathbb{R}$, and

$$\|G(z,0)\| = H(G(z,0),0) \leq q(z); \; z \in J.$$

(H_2) For every (z, w) in $J \times \mathbb{R}$, $G(z, w)$ is a nonempty convex and closed subset of $\mathbb{R}$.

Main results

(H_3) Assume that

$$p^* = \sup_{z\in J}\left|\frac{w_0}{\Gamma(\gamma)}\int_1^z \left(\ln\frac{z}{t}\right)^{\gamma-1} p(t)dt\right| < \infty,$$

$$q^* = \sup_{z\in J}\left|\frac{1}{\Gamma(\gamma)}\int_1^z \left(\ln\frac{z}{t}\right)^{\gamma-1} q(t)dt\right| < \infty,$$

$$p_\phi = \sup_{z\in J}\left|\frac{1}{\Gamma(\gamma)}\int_1^z \left(\ln\frac{z}{t}\right)^{\gamma-1} p(t)\left[\phi(t)-\phi(1)\right]dt\right| < 1,$$

Main results

Remarks

As observed there, G is lower semicontinuous, hence G admits continuous selection (Theorem 6).

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Theorem

Under assumptions $(H_1)-(H_3)$, the differential inclusion (1) has at least one solution $u = u(z)$ in the space $\mathcal{B}_\phi(J)$. Moreover, solutions of the differential inclusion (1) are globally attractive.

Main results

Proof.

Consider the multi-valued operator N defined on the space $\mathcal{B}_\phi(J)$ in the following way :

$$N : \mathcal{B}_\phi(J) \to \mathcal{P}\left(\mathcal{B}_\phi(J)\right);$$

such that, for each $w \in \mathcal{B}_\phi(J)$

$$(Nw)(z) = \{u \in \mathcal{B}_\phi(J) \mid u(z) = Lw(z)\},$$

□

Main results

Proof.

where

$$Lu(z) = w_1 + \frac{1}{\Gamma(r)} \int_1^z \left(\ln \frac{z}{t}\right)^{\gamma-1} g\left(t, w_0 + \int_1^t w(\tau)d\phi(\tau)\right) \frac{dt}{t},$$

and g is a selctor of G. For each function u in Nw, we have u is ϕ-continuous and bounded on J, so, the operator N is well defined. Using our hypothesis, w show that the operator $N : B_\zeta \longrightarrow \mathcal{P}(B_\zeta)$ is colsed, convex and

$$\psi(N\Omega) \leq p_\phi \psi(\Omega).$$

$\square$

Main results

Proof.

Then, by the Theorem 7 the operator N has at least one fixed point in B_ζ which is a solution of differential inclusion (1). Moreover, taking into account the fact that the set $Fix\ N \in \ker\ \psi$ (Remark 1) and the characterization of sets belonging to $\ker\ \psi$ (Remark 2), we conclude that all solutions of (1) are globally attractive. $\square$

Exemple

We consider the following differential inclusion

$$(7)\qquad \begin{cases} \mathcal{D}^{\gamma}\left(\frac{dw}{d\phi}\right)(z) \in \left[\frac{e^{-z}}{|u(z)|+z}, \frac{(z-1)e^{-z}}{|u(z)|+z}\right] \subset \mathbb{R};\ z > 1, \\ \frac{dw}{d\phi}(1) = w_0; \quad w(1) = w_1. \end{cases}$$

It is clear that (7) can be written as the differential inclusion (1), where $\phi(\tau) = \arctan([\tau])$ (the symbol $[\tau]$ indicates the integer value of τ). Let us show that conditions $(H_1) - (H_3)$ hold.

Exemple

For $z \in J$ and $u, v \in \mathbb{R}$, we have

$$H(F(z,u), F(z,v)) \leq (z-1)e^{-z}|u-v|.$$

So $\quad p(z) = (z-1)e^{-z}$, and we have

$$\|F(z,0)\| = \max\left\{\frac{e^{-z}}{z}, \frac{(z-1)}{z}e^{-z}\right\} = q(z).$$

Notice that the functions p, q are continuous and bounded on J. Next, for a fixed $z \in J$, we have $p^* < 1$ and this estimate show that p^* and q^* are finite quantities. Consequently from Theorem 10 the differential inclusion (7) has at least solution in the space $\mathcal{B}_\phi(J)$ and solutions of (7) are globally attractive.

THANKS