The 1st International Online Conference on Mathematics and Applications—A Celebration of the 10th Anniversary of Mathematics' Impact on Our Wellbeing

1–15 May 2023

Session 11 : Difference and Differential Equations

Existence and uniqueness of a solution of a Wentzell's problem with nonlinear delays

Presenter : **Ihaddadene Lila**
University of Sciences and Technology Houari Boumediene, Algiers, Algeria

Work plan

Introduction

Origin of the Wentzell's problem

Throughout history, the wave equation has known a great deal of work.
In our work, we are particularly interested in the wave equation with boundary conditions of the Wentzell type, which are characterized by the presence of differential operators of the same order as the main operator.
These problems are involved in the modeling of many phenomena:

- Mechanical like elasticity
- Physics such as diffusion processes or wave propagation

Origin of the Wentzell's problem

Wentzell's conditions are obtained by asymptotic methods from transmission problems, (**Lemrabet**. **K** [1]).
The following condition :

$$\partial_\nu u - \triangle_T u = g \qquad \text{on } \Gamma$$

for this equation

$$-\triangle u + u = f \qquad \text{in } \Omega$$

was first introduced by **Wentzell** (**Ventcel**) in **1959** [3], for diffusion processes.
It models the heat exchange of the body Ω with the surrounding environment in the presence of a thin film, very good conductor, on the surface of the body.

The delay effect

Delay is the property of a physical system by which the response to an applied force is retarded in its effect.
Whenever material, information, or energy is physically transmitted from one place to another, there is a delay present, a delay in the law of feedback modeling mechanical shift over time.
Delays so often occur in many :

- Physical problems
- Chemical, biological and economic phenomena

The model studied

We consider a wave equation with dynamical Wentzell type boundary conditions, two non linear dissipations and delay terms are localised in domain Ω and on part of boundary Γ_1, given by :

$$\begin{cases} u_{tt} - \Delta u + \mu_1 g_1(u_t) + \mu_2 g_1(u_t(t-\tau)) = 0, \text{ in } \Omega \times (0,\infty), \\ u = v, \text{ on } \Gamma \times (0,\infty), \\ u = 0, \text{ on } \Gamma_0 \times (0,\infty), \\ v_{tt} + \frac{\partial u}{\partial \nu} - \Delta_T v + \mu_1' g_2(v_t) + \mu_2' g_2(v_t(t-\tau)) = 0, \text{ on } \Gamma_1 \times (0,\infty). \end{cases} \tag{1}$$

Our objective is to show that this problem is well posed, that there is existence and uniqueness of a solution.

The model studied

Equipped with the following initial conditions

$$(u(0), v(0)) = (u_0, v_0), \text{ in } \Omega \times \Gamma, \tag{2}$$

$$(u_t(0), v_t(0)) = (u_1, v_1), \text{ in } \Omega \times \Gamma, \tag{3}$$

and

$$u_t(x, t-\tau) = f_{0_1}(x, t-\tau), \text{ in } \Omega \times (0, \tau), \tag{4}$$

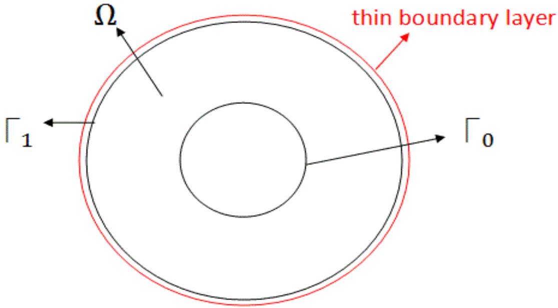
$$v_t(x, t-\tau) = f_{0_2}(x, t-\tau), \text{ on } \Gamma_1 \times (0, \tau). \tag{5}$$

The model studied

- Ω is a bounded open in $\mathbb{R}^n$, $(n \geq 2)$, with smooth boundary $\partial\Omega = \Gamma$, divided into two disjoint open subsets Γ_0 and Γ_1, such that $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\emptyset = \overline{\Gamma_0} \cup \overline{\Gamma_1}$.
- $\Delta_T v$ represents the tangentiel Laplacien.
- $\frac{\partial u}{\partial \nu}$ is the normal derivative of u where ν represents the normal unit field to Γ, outward to Ω.
- The terms $g_1(u_t(x, t-\tau))$ and $g_2(v_t(x, t-\tau))$ describe the delays on the nonlinear frictional dissipations $g_1(u_t)$ and $g_2(v_t)$, on Ω and Γ_1, respectively.
- μ_1, μ_1', μ_2 and μ_2' are positive real numbers.
- $\tau > 0$ is a time delay.

The model studied

This model describes vibrations of a flexible body with a thin boundary layer of high rigidity on its boundary Γ_1.



Figure

System transformation (1)-(5) :

We consider the following change of functions :

$$\begin{cases} z_1(x,\rho,t) = u_t(x, t-\rho\tau), & x \in \Omega, \quad \rho \in (0,1), \quad t > 0, \\ z_2(x,\rho,t) = v_t(x, t-\rho\tau), & x \in \Gamma_1, \quad \rho \in (0,1), \quad t > 0. \end{cases}$$

Therefore, the system (1)-(5) is equivalent to :

System transformation (1)-(5) :

$$\begin{cases} u_{tt} - \Delta u + \mu_1 g_1(u_t) + \mu_2 g_1(z_1(x,1,t)) = 0, \text{ in } \Omega \times (0,\infty), \\ v_{tt} + \partial_\nu u - \Delta_T v + \mu_1' g_2(v_t) + \mu_2' g_2(z_2(x,1,t)) = 0, \text{ on } \Gamma_1 \times (0,\infty), \\ \tau (z_1)_t(x,\rho,t) + (z_1)_\rho(x,\rho,t) = 0, \text{ in } \Omega \times (0,1) \times (0,\infty), \\ \tau (z_2)_t(x,\rho,t) + (z_2)_\rho(x,\rho,t) = 0, \text{ on } \Gamma_1 \times (0,1) \times (0,\infty), \\ u = v, \text{ on } \Gamma \times \mathbb{R}^+, \\ u = 0, \text{ on } \Gamma_0 \times \mathbb{R}^+, \\ z_1(x,0,t) = u_t(x,t), \text{ in } \Omega \times \mathbb{R}^+, \\ z_2(x,0,t) = v_t(x,t), \text{ on } \Gamma_1 \times \mathbb{R}^+, \\ (u(0), v(0)) = (u_0, v_0), \text{ in } \Omega \times \Gamma, \\ (u_t(0), v_t(0)) = (u_1, v_1), \text{ in } \Omega \times \Gamma, \\ z_1(x,\rho,0) = f_{0_1}(x, -\rho\tau), \text{ in } \Omega \times (0,1), \\ z_2(x,\rho,0) = f_{0_2}(x, -\rho\tau), \text{ on } \Gamma_1 \times (0,1). \end{cases} \tag{6}$$

Assumptions on the damping and delay functions gi for i = 1,2:

(A1) $g_i : \mathbb{R} \longrightarrow \mathbb{R}$ is an odd non decreasing function of the class $\mathcal{C}^1(\mathbb{R})$ such that there exist r,(sufficiently small), c_i, C_i, c, α_1, and $\alpha_2 > 0$ for $i = 1, 2$ and a convex, increasing function: $H : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ of the class $\mathcal{C}^1(\mathbb{R}_+) \cap \mathcal{C}^2(]0, \infty[)$, satisfying : $H(0) = 0$ and H linear on $[0, r]$ or $(H'(0) = 0$ and $H'' > 0$ on $]0, r])$, such that

$$c_i\,|s| \leq |g_i(s)| \leq C_i\,|s| \quad \text{if } |s| \geq r, \tag{7}$$

$$s^2 + g_i^2(s) \leq H^{-1}(s g_i(s)) \quad \text{if } |s| \leq r, \tag{8}$$

Assumptions on the damping and delay functions gi for i = 1,2:

$$\left|g_i'(s)\right| \leq c, \tag{9}$$

$$\alpha_1 s g_i(s) \leq G_i(s) \leq \alpha_2 s g_i(s), \tag{10}$$

where

$$G_i(s) = \int_0^s g_i(y)dy.$$

(A2) $\alpha_2\mu_2 < \alpha_1\mu_1$ and $\alpha_2\mu_2' < \alpha_1\mu_1'$.

The energy of the problem (6):

We define the energy associated with the solution of the problem (6) by :

$$E(t)=\frac{1}{2}\left\|u_t\right\|^2+\frac{1}{2}\left\|\nabla u\right\|^2+\frac{1}{2}\left\|v_t\right\|_{\Gamma_1}^2+\frac{1}{2}\left\|\nabla_T v\right\|_{\Gamma_1}^2$$

$$+\xi\int_{\Omega}\left(\int_0^1 G_1(z_1(x,\rho,t))d\rho\right)dx+\zeta\int_{\Gamma_1}\left(\int_0^1 G_2(z_2(x,\rho,t))d\rho\right)d\sigma, \tag{11}$$

where ξ and ζ are strictly positive constants, such that

$$\tau\frac{\mu_2(1-\alpha_1)}{\alpha_1}<\xi<\tau\frac{\mu_1-\alpha_2\mu_2}{\alpha_2}, \tag{12}$$

$$\tau\frac{\mu_2'(1-\alpha_1)}{\alpha_1}<\zeta<\tau\frac{\mu_1'-\alpha_2\mu_2'}{\alpha_2}. \tag{13}$$

Energy decay

The following lemma shows that the system (6) is dissipative.

Lemma

Let (u, v, z_1, z_2) be a solution of the problem (6). Then, there exist positive constants a_1, a_2, a_3 and a_4 such that for all $t \geq 0$:

$$\begin{aligned} E'(t) &\leq -a_1 \int_\Omega u_t g_1(u_t)dx - a_2 \int_{\Gamma_1} v_t g_2(v_t)d\sigma \\ &\quad -a_3 \int_\Omega z_1(x,1,t))g_1(z_1(x,1,t))dx \\ &\quad -a_4 \int_{\Gamma_1} z_2(x,1,t))g_2(z_2(x,1,t))d\sigma \\ &\leq 0, \end{aligned} \tag{14}$$

Energy decay

where $a_1 = \left(\mu_1 - \frac{\xi}{\tau}\alpha_2 - \mu_2\alpha_2\right)$, $a_2 = \left(\mu_1' - \frac{\zeta}{\tau}\alpha_2 - \mu_2'\alpha_2\right)$,
$a_3 = \left(\alpha_1\frac{\xi}{\tau} - \mu_2(1-\alpha_1)\right)$ and $a_4 = \left(\alpha_1\frac{\zeta}{\tau} - \mu_2'(1-\alpha_1)\right)$.

Existence and uniqueness result of a solution

Existence and uniqueness theorem

We introduce the following set:

$$H^1_{\Gamma_0}(\Omega) = \left\{ u \in H^1(\Omega) \,/\, \ u|_{\Gamma_0} = 0 \right\},$$

endow $H^1_{\Gamma_0}(\Omega)$ with the Hilbert structure induced by $H^1(\Omega)$.
Now, we state the existence and uniqueness theorem :

Theorem

Let $(u_0, u_1, v_0, v_1) \in$
$H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)) \times H^1_{\Gamma_0}(\Omega) \times H^2(\Gamma_1) \times H^1(\Gamma_1)$,
$f_{0_1} \in H^1_{\Gamma_0}\left(\Omega; H^1(0,1)\right)$ *and* $f_{0_2} \in H^1\left(\Gamma_1; H^1(0,1)\right)$, *satisfy the following compatibility condition :*

$$\begin{cases} \partial_\nu u_0 - \Delta_T v_0 + \mu_1' g_2(v_1) = 0 & \text{on} \quad \Gamma_1, \\ f_{0_1}(.,0) = u_t & \text{in} \quad \Omega, \\ f_{0_2}(.,0) = v_t & \text{on} \quad \Gamma_1. \end{cases} \tag{15}$$

Existence and uniqueness theorem

We suppose that (**A1**) and (**A2**) hold, then problem (6) possesses a unique global weak solution satisfying for $T > 0$:

$$\begin{aligned}(u, u_t, u_{tt}) &\in L^\infty(0, T; \left[H^1_{\Gamma_0}(\Omega)\right]^2 \times L^2(\Omega)),\\ (v, v_t, v_{tt}) &\in L^\infty(0, T; \left[H^1(\Gamma_1)\right]^2 \times L^2(\Gamma_1)).\end{aligned}$$

Proof of Theorem

We use the Faedo-Galerkin's method.
Let us define the approximations u^n, v^n, z_1^n and z_2^n by

$$\begin{aligned} u^n(t) &= \sum_{i=1}^{n} a_i^n(t) w_i, \; v^n(t) = \sum_{i=1}^{n} b_i^n(t) \widetilde{w}_i, \; z_1^n(t) = \sum_{i=1}^{n} c_i^n(t) \phi_i, \\ z_2^n(t) &= \sum_{i=1}^{n} d_i^n(t) \widetilde{\phi}_i, \end{aligned}$$

where a_i^n, b_i^n, c_i^n and d_i^n are from the class $\mathcal{C}^2$ and $\{w_i\}_{1\leq i\leq n}$, $\{\widetilde{w}_i\}_{1\leq i\leq n}$, $\{\phi_i\}_{1\leq i\leq n}$ and $\{\widetilde{\phi}_i\}_{1\leq i\leq n}$ are basis in the spaces $H^2(\Omega)\cap H^1_{\Gamma_0}(\Omega)$, $H^2(\Gamma_1)\cap H^1(\Gamma_1)$, $L^2(\Omega\times(0,1))$ and $L^2(\Gamma_1\times(0,1))$, respectively,

Proof of Theorem

and verifying the following differential equations:

$$\begin{aligned}(u_{tt}^n, w_i) + (\nabla u^n, \nabla w_i) + \mu_1(g_1(u_t^n), w_i) + \mu_2(g_1(z_1^n(x,1,t)), w_i) \\ +(v_{tt}^n, \widetilde{w}_i)_{\Gamma_1} + (\nabla_T v^n, \nabla_T \widetilde{w}_i)_{\Gamma_1} + \mu_1'(g_2(v_t^n), \widetilde{w}_i)_{\Gamma_1} \\ +\mu_2'(g_2(z_2^n(x,1,t)), \widetilde{w}_i)_{\Gamma_1} = 0, \qquad 1 \le i \le n,\end{aligned} \tag{16}$$

$$\int_\Omega \int_0^1 (\tau z_{1_t}^n + z_{1_\rho}^n)\phi_i d\rho dx = 0, \qquad 1 \le i \le n \tag{17}$$

and

$$\int_{\Gamma_1} \int_0^1 (\tau z_{2_t}^n + z_{2_\rho}^n)\widetilde{\phi}_i d\rho d\sigma = 0, \qquad 1 \le i \le n, \tag{18}$$

Proof of Theorem

with initial data:

$$\begin{cases} u^n(0) = u_0^n = \sum_{i=1}^n a_i^n(0) w_i \longrightarrow u_0 & \text{in} \quad \left(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right), \\ u_t^n(0) = u_1^n = \sum_{i=1}^n (a_i^n)_t(0) w_i \longrightarrow u_1 & \text{in} \quad H^1_{\Gamma_0}(\Omega), \\ v^n(0) = v_0^n = \sum_{i=1}^n b_i^n(0) \widetilde{w}_i \longrightarrow v_0 & \text{in} \quad \left(H^2(\Gamma_1) \cap H^1(\Gamma_1)\right), \\ v_t^n(0) = v_1^n = \sum_{i=1}^n (b_i^n)_t(0) \widetilde{w}_i \longrightarrow v_1 & \text{in} \quad H^1(\Gamma_1), \\ z_1^n(\rho, 0) = z_{0_1}^n = \sum_{i=1}^n c_i^n(0) \phi_i \longrightarrow f_{0_1} & \text{in} \quad H^1_{\Gamma_0}\left(\Omega; H^1(0,1)\right), \\ z_2^n(\rho, 0) = z_{0_2}^n = \sum_{i=1}^n d_i^n(0) \widetilde{\phi}_i \longrightarrow f_{0_2} & \text{in} \quad H^1\left(\Gamma_1; H^1(0,1)\right). \end{cases} \tag{19}$$

Proof of Theorem

The local existence of solutions of the problem (16)-(19) is standard by the theory of ordinary differential equations, we can conclude that there is a $t_n > 0$ such that in $[0, t_n]$, the problem (16)-(19) has a unique local solution which can be extended to a maximal interval $[0, T]$ (with $0 < T \leq \infty$) by Zorn's lemma, since the nonlinear terms in (16) are locally Lipschitz continuous.
We can utilize a standard compactness argument for the limiting procedure and it suffices to derive some a priori estimates for (u^n, v^n, z_1^n, z_2^n).

Proof of Theorem

First estimate

We obtain, for any $T > 0$

$$u^n \text{ is bounded in } L^\infty\left(0, T; H^1_{\Gamma_0}(\Omega)\right), \tag{20}$$

$$v^n \text{ is bounded in } L^\infty\left(0, T; H^1(\Gamma_1)\right), \tag{21}$$

$$u^n_t \text{ is bounded in } L^\infty\left(0, T; L^2(\Omega)\right), \tag{22}$$

$$v^n_t \text{ is bounded in } L^\infty\left(0, T; L^2(\Gamma_1)\right), \tag{23}$$

$$u^n_t g_1(u^n_t) \text{ is bounded in } L^1(\Omega \times (0, T)), \tag{24}$$

$$v^n_t g_2(v^n_t) \text{ is bounded in } L^1(\Gamma_1 \times (0, T)), \tag{25}$$

Proof of Theorem

First estimate

$$G_1(z_1^n)) \text{ is bounded in } L^\infty\left(0,T;L^1(\Omega\times(0,1))\right), \tag{26}$$

$$G_2(z_2^n)) \text{ is bounded in } L^\infty\left(0,T;L^1(\Gamma_1\times(0,1))\right), \tag{27}$$

$$z_1^n(x,1,t)g_1(z_1^n(x,1,t)) \text{ is bounded in } L^1(\Omega\times(0,T)), \tag{28}$$

$$z_2^n(x,1,t)g_2(z_2^n(x,1,t)) \text{ is bounded in } L^1(\Gamma_1\times(0,T)). \tag{29}$$

Proof of Theorem

Second estimate

We obtain

$$u_t^n \text{ is bounded in } L^\infty\left(0, T; H^1_{\Gamma_0}(\Omega)\right), \tag{30}$$

$$v_t^n \text{ is bounded in } L^\infty\left(0, T; H^1(\Gamma_1)\right), \tag{31}$$

$$u_{tt}^n \text{ is bounded in } L^\infty\left(0, T; L^2(\Omega)\right), \tag{32}$$

$$v_{tt}^n \text{ is bounded in } L^\infty\left(0, T; L^2(\Gamma_1)\right), \tag{33}$$

Proof of Theorem

Second estimate

$$z_{1_t}^n \text{ is bounded in } L^\infty(0,T;L^2(\Omega\times(0,1)), \tag{34}$$

$$z_{2_t}^n \text{ is bounded in } L^\infty(0,T;L^2(\Gamma_1\times(0,1)). \tag{35}$$

Proof of Theorem

Estimate for $\mathbf{z}_1^n$ and $\mathbf{z}_2^n$

We obtain

$$z_1^n \text{ is bounded in } L^\infty\left(0, T; H^1_{\Gamma_0}\left(\Omega; L^2\left(0,1\right)\right)\right), \tag{36}$$

$$z_2^n \text{ is bounded in } L^\infty\left(0, T; H^1\left(\Gamma_1; L^2\left(0,1\right)\right)\right). \tag{37}$$

Proof of Theorem

The passing to the limit

Applying Dunford-Petti's theorem, we conclude from (20)-(37) that there exists subsequences of $(u^n)_n$, $(v^n)_n$, $(z_1^n)_n$ and $(z_2^n)_n$ which we still denote by $(u^n)_n$, $(v^n)_n$, $(z_1^n)_n$ and $(z_2^n)_n$ respectively, such that

$$\left(u^n, u_t^n, u_{tt}^n\right) \rightharpoonup \left(u, u_t, u_{tt}\right) \text{ weakly-star in } L^\infty\left(0, T; \left[H^1_{\Gamma_0}(\Omega)\right]^2 \times L^2(\Omega)\right) \tag{38}$$

$$\left(v^n, v_t^n, v_{tt}^n\right) \rightharpoonup \left(v, v_t, v_{tt}\right) \text{ weakly-star in } L^\infty\left(0, T; \left[H^1(\Gamma_1)\right]^2 \times L^2(\Gamma_1)\right) \tag{39}$$

$$g_1(u_t^n) \rightharpoonup \chi_1 \text{ weakly-star in } L^2((0, T) \times \Omega),$$

$$g_2(v_t^n) \rightharpoonup \chi_2 \text{ weakly-star in } L^2((0, T) \times \Gamma_1),$$

Proof of Theorem

The passing to the limit

$$z_1^n \rightharpoonup z_1 \text{ weakly-star in } L^\infty\left(0, T; H^1_{\Gamma_0}\left(\Omega; L^2\left(0,1\right)\right)\right), \tag{40}$$

$$z_2^n \rightharpoonup z_2 \text{ weakly-star in } L^\infty\left(0, T; H^1\left(\Gamma_1; L^2\left(0,1\right)\right)\right), \tag{41}$$

$$z_{1_t}^n \rightharpoonup z_{1_t} \text{ weakly-star in } L^\infty(0, T; L^2(\Omega \times (0,1))), \tag{42}$$

$$z_{2_t}^n \rightharpoonup z_{2_t}^n \text{ weakly-star in } L^\infty(0, T; L^2(\Gamma_1 \times (0,1))), \tag{43}$$

$$g_1(z_1^n(x,1,t)) \rightharpoonup \Psi_1 \text{ weakly-star in } L^2((0,T) \times \Omega),$$

$$g_2(z_2^n(x,1,t)) \rightharpoonup \Psi_2 \text{ weakly-star in } L^2((0,T) \times \Gamma_1).$$

Proof of Theorem

The passing to the limit

Next, thanks to Aubin-Lions's theorem, we arrive at

$$u^n \longrightarrow u \text{ strongly in } L^2\left(0, T; L^2(\Omega)\right), \tag{44}$$

$$u_t^n \longrightarrow u_t \text{ strongly in } L^2\left(0, T; L^2(\Omega)\right), \tag{45}$$

$$v^n \longrightarrow v \text{ strongly in } L^2\left(0, T; L^2(\Gamma_1)\right), \tag{46}$$

$$v_t^n \longrightarrow v_t \text{ strongly in } L^2\left(0, T; L^2(\Gamma_1)\right), \tag{47}$$

$$z_1^n \longrightarrow z_1 \text{ strongly in } L^2\left(\Omega \times (0,1) \times (0, T)\right), \tag{48}$$

$$z_2^n \longrightarrow z_2 \text{ strongly in } L^2\left(\Gamma_1 \times (0,1) \times (0, T)\right). \tag{49}$$

Proof of Theorem

Analysis of the nonlinear terms

We obtain

$$\begin{cases} g_1\left(u_t^n\right) \rightharpoonup g_1\left(u_t\right) \text{ weakly in } L^2(\Omega\times(0,T)), \\ g_2\left(v_t^n\right) \rightharpoonup g_2\left(v_t\right) \text{ weakly in } L^2(\Gamma_1\times(0,T)). \end{cases} \tag{50}$$

And

$$\begin{cases} g_1(z_1^n(x,1,t)) \rightharpoonup g_1\left(z_1(x,1,t)\right) \text{ weakly in } L^2(\Omega\times(0,T)), \\ g_2(z_2^n(x,1,t)) \rightharpoonup g_2\left(z_2(x,1,t)\right) \text{ weakly in } L^2(\Gamma_1\times(0,T)). \end{cases} \tag{51}$$

Proof of Theorem

Uniqueness of solution

By reasoning by absurd, i.e., supposing that there exist two solutions (u, v, z_1, z_2) and $(\tilde{u}, \tilde{v}, \tilde{z}_1, \tilde{z}_2)$ of problem (6), with $(U, V, Z_1, Z_2) = (u, v, z_1, z_2) - (\tilde{u}, \tilde{v}, \tilde{z}_1, \tilde{z}_2)$.

Then, we will arrive at $(U, V, Z_1, Z_2) = 0$, hence the uniqueness.

This finish the proof of Theorem. ■

Conclusion

Conclusion

We studied the existence and uniqueness of a solution thanks to the Faedo Galerkin's approximation, of a coupled wave/Wentzell system, in the presence of two nonlinear frictional dampings and two nonlinear delay terms, localized inside Ω and on the part Γ_1 of its boundary.

References

Lemrabet. K., *Etude de divers problèmes aux limites de Ventcel d'origine physique ou mécanique dans des domaines non réguliers*, USTHB, Alger, 1987.

Nicaise. S., **Pignotti. C.**, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks*, SIAM Journal on Control and Optimization, (2006), 45: 1561-1585.

Wentzell (**Ventcel**). **A. D.**, *On boundary conditions for multidimentional diffusion processes*, Theory of Probability and Its Applications, 4 (1959), 164-177.

Thank you for your attention