computer sciences and mathematics forum

MDPI

Proceeding Paper

New Aspects in the Theory of Complete Hypergroups †

Irina Cristea

Center for Information Technologies and Applied Mathematics, University of Nova Gorica, Vipavska cesta 13, 5000 Nova Gorica, Slovenia; irina.cristea@ung.si

† Presented at the 1st International Online Conference on Mathematics and Applications; Available online: https://iocma2023.sciforum.net/.

Abstract: The aim of this note is to review the most important properties and applications of the complete hypergroups. We will focus on the reversibility, regularity and reducibility properties, on the class equation and the commutativity degree of the complete hypergroups, as well as on the Euler's totient function defined on this kind oh hypergroups.

Keywords: complete hypergroup; reversibility; regularity; reducibility; class equation; commutativity degree; Euler's totient function

check for updates

Citation: Cristea, I. New Aspects in the Theory of Complete Hypergroups. *Comput. Sci. Math. Forum* **2023**, *1*, 0. https://doi.org/

Academic Editor: Adolfo Ballester-Bolinches

Published: 28 April 2023

1. Introduction and preliminaries

Starting with 1934, when F. Marty introduced the hypergroup as a new algebraic structure extending the classical one of group, and satisfying the same properties, i.e., the associativity and reproductivity, the theory of hypercompositional structures (called also hyperstructure theory) has experienced a rapid growth, succeeding to impose itself as a branch of Abstract Algebra. Nowadays this theory offers a strong background for studies in algebraic geometry [1], number theory [2], automata theory [3], graph theory [4], matroids theory [5], association schemes [6], to list just some of the research fields where hypercompositional structures are deeply involved. These structures are non-empty sets endowed with at list one hyperoperation, i.e., a multivalued operation resulting in a subset of the underlying set, usually denoted as $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$, where $(P)^*(H)$ denotes the set of non-empty subsets of H. A non-empty set H equipped with a hyperoperation that satisfies: (i) the associativity: $(x \circ y) \circ z = x \circ (y \circ z)$ for any $x, y, z \in H$, where $(x \circ y) \circ z = \bigcup_{u \in x \circ y} u \circ z$ and $x \circ (y \circ z) = \bigcup_{v \in y \circ z} x \circ v$, and (ii) 0the reproductivity: $x \circ H = H = H \circ x$, for any $x \in H$, is called a *hypergroup*. One particular type of hypergroups is represented by the *complete hypergroups*, introduced in 1970 by Koskas [7], based on the notions of complete part and complete closure. Given a hypergroup $(H, \circ)$, a non-empty subset A of H is called a complete part if, for any $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in H$ such that $(x_1 \circ \cdots \circ x_n) \cap A \neq \emptyset$, it follows that $x_1 \circ \cdots \circ x_n \subseteq A$. The intersection of all complete parts of H containing A is then called the complete closure $\mathcal{C}(A)$ of A in H. A hypergroup $(H, \circ)$ is complete if, for any $(x, y) \in H^2$, $\mathcal{C}(c \circ y) = x \circ y$. The complete hypergroups are easily described with the help of groups as explained in the following theorem, called the characterization theorem.

Theorem 1. [8] *A complete hypergroup $(H, \circ)$ can be represented as the union $H = \bigcup_{g \in G} A_g$ of its non-empty subsets A_g, with $g \in G$, where:*

1. *$(G, \cdot)$ is an arbitrary group.*
2. *For any $g_1 \neq g_2 \in G$, the subsets A_{g_1} and A_{g_2} are disjoint.*
3. *The hyperoperation on H is defined by the rule: if $(a, b) \in A_{g_1} \times A_{g_2}$, then $a \circ b = A_{g_1 \cdot g_2}$.*

We refer to G as the underlying group of the complete hypergroup H. Notice that if G and H have the same cardinality, then they coincide and the complete hypergroup H is a group. Conversely, any group can be seen as a complete hypergroup. These are particular cases, called improper complete hypergroups, that we will exclude from our study, where we will deal only with proper complete hypergroups.

Example 1. *Let $G = S_3 = \langle (12), (123) \rangle$ be the permutation group with 6 elements generated by the transposition (12) and the 3-cycle (123), i.e., $G = \{e, (12), (23), (31), (123), (321)\}$ and $(H, \circ)$ a complete hypergroup with 9 elements having the underlying group G and given by the following partition: $A_e = \{a_0\}$, $A_{(12)} = \{a_1, a_2\}$, $A_{(23)} = \{a_3, a_4\}$, $A_{(31)} = \{a_5, a_6\}$, $A_{(123)} = \{a_7\}$, $A_{(321)} = \{a_8\}$. Based on the characterization theorem, the Cayley table of the complete hypergroup H is the following one:*

$\circ$	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
a_0	a_0	a_1, a_2	a_1, a_2	a_3, a_4	a_3, a_4	a_5, a_6	a_5, a_6	a_7	a_8
a_1	a_1, a_2	a_0	a_0	a_7	a_7	a_8	a_8	a_3, a_4	a_5, a_6
a_2	a_1, a_2	a_0	a_0	a_7	a_7	a_8	a_8	a_3, a_4	a_5, a_6
a_3	a_3, a_4	a_8	a_8	a_0	a_0	a_7	a_7	a_5, a_6	a_1, a_2
a_4	a_3, a_4	a_8	a_8	a_0	a_0	a_7	a_7	a_5, a_6	a_1, a_2
a_5	a_5, a_6	a_7	a_7	a_8	a_8	a_0	a_0	a_1, a_2	a_3, a_4
a_6	a_5, a_6	a_7	a_7	a_8	a_8	a_0	a_0	a_1, a_2	a_3, a_4
a_7	a_7	a_5, a_6	a_5, a_6	a_1, a_2	a_1, a_2	a_3, a_4	a_3, a_4	a_8	a_0
a_8	a_8	a_3, a_4	a_3, a_4	a_5, a_6	a_5, a_6	a_1, a_2	a_1, a_2	a_0	a_7

Example 2. *Considering the same group G as in Example 1 and a complete hypergroup H of the same cardinality 9, but with a different partition, i.e., $A_e = \{a_0, a_1\}$, $A_{(12)} = \{a_2\}$, $A_{(23)} = \{a_3\}$, $A_{(31)} = \{a_4\}$, $A_{(123)} = \{a_5, a_6\}$, $A_{(321)} = \{a_7, a_8\}$, we get the following Cayley table for H:*

$\circ$	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
a_0	a_0, a_1	a_0, a_1	a_2	a_3	a_4	a_5, a_6	a_5, a_6	a_7, a_8	a_7, a_8
a_1	a_0, a_1	a_0, a_1	a_2	a_3	a_4	a_5, a_6	a_5, a_6	a_7, a_8	a_7, a_8
a_2	a_2	a_2	a_0, a_1	a_5, a_6	a_7, a_8	a_3	a_3	a_4	a_4
a_3	a_3	a_3	a_7, a_8	a_0, a_1	a_5, a_6	a_4	a_4	a_2	a_2
a_4	a_4	a_4	a_5, a_6	a_7, a_8	a_0, a_1	a_2	a_2	a_3	a_3
a_5	a_5, a_6	a_5, a_6	a_4	a_2	a_3	a_7, a_8	a_7, a_8	a_0, a_1	a_0, a_1
a_6	a_5, a_6	a_5, a_6	a_4	a_2	a_3	a_7, a_8	a_7, a_8	a_0, a_1	a_0, a_1
a_7	a_7, a_8	a_7, a_8	a_3	a_4	a_2	a_0, a_1	a_0, a_1	a_5, a_6	a_5, a_6
a_8	a_7, a_8	a_7, a_8	a_3	a_4	a_2	a_0, a_1	a_0, a_1	a_5, a_6	a_5, a_6

Notice that using the same underlying group but different partitions, we may get non-isomorphic complete hypergroups of the same cardinality.

The aim of this presentation is to gather the main properties of complete hypergroups, with an emphasis on those studied in the last years. We believe that an overview on this theory, fixing the notations and terminology, will open new lines of research on complete hypergroups or other topics related to them.

2. Properties of Complete Hypergroups

For a complete understanding of this topic, the reader is refereed to the books [8,9] and overview articles [10,11].

2.1. Reversibility, Regularity and Reducibility Properties

One main difference between groups and hypergroups stays in the existence of the identity elements. In a group, an identity element always exists and it is unique, while in a hypergroup it may exist or not, it may be unique or not. The same property applies on inverses. Let us recall their definitions.

An element e in a hypergroup $(H, \circ)$ is called a left (or right) identity, if $a \in e \circ a$ (or $a \in a \circ e$) for all $a \in H$. If , for all $a \in H$, we have $a \in a \circ e \cap e \circ a$, then a is called a bilateral identity (or simply, an identity). An element $a^{-1} \in H$ is called a left (or right) inverse of $a \in H$, if there exists an identity $e \in H$ such that $e \in a^{-1} \circ a$ (or $e \in a \circ a^{-1}$). If a^{-1} is a left and right inverse for $a \in H$, then it is called an inverse of a and a is an invertible element.

A hypergroup $(H, \circ)$ is called regular if it has at least one bilateral identity and each element has at least one inverse. A regular hypergroup is called reversible if for each $a, b, c \in H$ such that $a \in b \circ c$ it follows that $b \in c \circ a^{-1}$ and $c \in b^{-1} \circ a$.

The natural bridge between the classical algebraic structures and the related hypercompositional structures is represented by the fundamental relations. There exist two groups of such relations: the first one is composed with those relations that connect groups with hypergroups, rings with hyperrings and so on, while the second one refers to the equivalences defined by Jantosciak [12] in order to obtain reduced hypergroups. Within the first group, we recall the equivalence β defined as $\beta = \bigcup_{n \geq 1} \beta_n$, where β_1 is the diagonal relation on H and β_n is obtained by the rule $a\beta_n b$ if and only if there exist $x_1 \ldots, x_n \in H$ such that $\{a, b\} \subseteq x_1 \circ \cdots \circ x_n$. It is well known that in the case of a hypergroup, β is the samllest equivalence such that the quotient H/β is a group. The heart of a hypergrou is then the set $\omega_H = \{x \in H \mid \varphi_H(x) = 1\}$, where $\varphi : H \longrightarrow H/\beta$.

In the following result we will summarize the main properties of the complete hypergroups related to the above mentioned concepts.

Theorem 2. *[13] For any complete hypergroup $(H, \circ)$ with the underlying group G, i.e., $H = \bigcup_{g \in G} A_g$, the following properties hold:*

(1) *The heart ω_H is the set of all bilateral identities of H. In particular, $\omega_H = A_e$, where e is the identity of the group G.*

(2) *H is a reversible and regular hypergroup.*

Unlike what happens in group theory, two elements in a hypergroup may play interchangable roles with respect to the hyperoperation, that mathematically are described by the following three equivalences, called fundamental relations by Jantosciak [12]. In a hypergroup $(H, \circ)$ two elements x and y are called

- operationally equivalent, denoted $x \sim_o y$, if $x \circ a = y \circ a$ and $a \circ x = a \circ y$ for any $a \in H$;
- inseparable, denoted $x \sim_i y$, if $x \in a \circ b$ if and only if $y \in a \circ b$ for $a, b \in H$;
- essentially indistinguishable, denoted $x \sim_e y$, if $x \sim_o y$ and $x \sim_i y$.

A hypergroup is then called reduced if the equivalence class of any element in H is a singleton. For any proper complete hypergroup, the essentially indistinguishable relation has the following interpretation: $x \sim_e y \iff \exists g \in G : x, y \in A_g$. Considering Example 1, one notice that the equivalence classes with respect to the fundamental relation $\sim_e$ are indeed the sets A_σ, with $\sigma \in S_3$. We can conlude with the following property.

Theorem 3. *[13] Any proper complete hypergroup is not reduced.*

2.2. The Class Equation

Two elements a and b are conjugated if there exists $g \in G$ such that $ga = bg$. This is an equivalence relation on G and the equivalence class of a, called the conjugacy class, is denoted by $[a]$. Knowing that $k(G)$ is the number of the distinct conjugacy classes of the elements of G and $Z(G)$ is the centre of G, we may write the class equation as follows:

$$|G| = |Z(G)| + \sum_{i=|Z(G)|+1}^{k(G)} |[x_i]|,$$

where $G = \bigcup_{i=1}^{k(G)} [x_i]$.

In order to get a similar equation on a hypergroup, we first define the conjugation relation using the complete closure of a set. Two elements a and b in a hypergroup H are conjugated if there exists $c \in H$ such that $\mathcal{C}(c \circ a) \cap \mathcal{C}(b \circ c) \neq \emptyset$ and we denote it by $a \sim_H b$. In a complete hypergroup, this relation reduces to a simpler form, very similar to the one defined on groups, and indeed $a \sim_H b \iff \exists c \in H : c \circ a = b \circ c$. Based now on the characterization theorem of a complete hypergroup written as $H = \bigcup_{g \in G} A_g$, two elements $a \in A_{g_1}$ and $b \in A_{g_2}$ are conjugated if and only if g_1 and g_2 are conjugated in G. Thus, the number $k(H)$ of the distinct conjugacy classes in the finite complete hypergroup H is equal with the number $k(G)$ of the distinct conjugacy classes of the underlying group G and the following theorem holds.

Theorem 4. *[14] The class equation for a finite complete hypergroup H has the form*

$$|H| = |\omega_H| + \sum_{a \notin \omega_H} |[a]|.$$

2.3. The Commutativity Degree

One of the famous arithmetic functions defined on a group is the commutativity degree, expressed as the probability that two distinct elements commute in the group. Defining $c(G) = \{(x, y) \in G^2 \mid xy = yx\}$, the commutativity degree is the number

$$d(G) = \frac{|c(G)|}{|G|^2}.$$

Extending this definition to complete hypergroups, we may introduce the commutativity degree on a hypergroup as follows:

$$d(H) = \frac{|\{(a, b) \in H^2 \mid \exists g_i, g_j \in G, a \in A_{g_i}, b \in A_{g_j}, g_i g_j = g_j g_i\}|}{|H|^2},$$

that can be also expressed using the conjugacy classes of the elements in H. If $C_G(g) = \{h \in G \mid gh = hg\}$ is the centralizer of the element g in G, then the centralizer of $x \in H$ has the form $C_H(x) = \{y \in H \mid x \circ y = y \circ x\} = \bigcup_{g \in C_G(g_x)} A_g$, where for $x \in H$, there exists an unique $g_x \in G$ such that $x \in A_{g_x}$.

Theorem 5. *[14] Let $H = \bigcup_{g \in G} A_g$ be a finite complete hypergroup with the underlying group G such that $|C_H(x)| = |C_H(y)|$, for any y in the conjugacy class $[x]$ of x. Then, the commutativity degree of H is*

$$d(H) = \frac{\sum_{i=1}^{k(H)} |[x_i]| \cdot |C_H(x_i)|}{|H|^2},$$

where $k(H)$ denotes the number of distinct conjugacy classes of the elements in H.

The condition expressed in the hypothesis of Theorem 5 is a necessary one in order to have the formula given before.

Example 3. *Continuing with Example 2 and calculating the conjugacy classes of the elements in the group S_3, we obtain $[e] = \{e\}$, $[(12)] = \{(12), (23), (31)\}$ and $[(123)] = \{(123), (321)\}$, while their centralizers are $C_{S_3}(e) = S_3$, $C_{S_3}((12)) = \{e, (12)\}$, $C_{S_3}((23)) = \{e, (23)\}$, $C_{S_3}((31)) = \{e, (31)\}$, $C_{S_3}((123)) = \{e, (123), (321)\}$. Let us calculate now the conjugacy classes of the elements of the complete hypergroup H: $[a_0] = [a_1] = A_e$, $[a_2] = [a_3] = [a_4] = A_{(12)} \cup A_{(23)} \cup A_{(31)} = \{a_2, a_3, a_4\}$ and $[a_5] = [a_6] = [a_7] = [a_8] = A_{(123)} \cup A_{(321)} = \{a_5, a_6, a_7, a_8\}$. Their centralizers are: $C_H(a_0) = C_H(a_1) = H$, $C_H(a_2) = \{a_0, a_1, a_2\}$, $C_H(a_3) = \{a_0, a_1, a_3\}$, $C_H(a_4) = \{a_0, a_1, a_4\}$ and finally $C_H(a_5) = C_H(a_6) = C_H(a_7) =$*

$C_H(a_8) = \{a_0, a_1, a_5, a_6, a_7, a_8\}$. *We notice immediately that the condition in Theorem 5 is fulfilled. Based on this data, the computation of the commutativity degree is then*

$$d(H) = \frac{|[a_0]| \cdot |C_H(a_0)| + |[a_2]| \cdot |C_H(a_2)| + |[a_5]| \cdot |C_H(a_5)|}{81} = \frac{2 \cdot 9 + 3 \cdot 3 + 4 \cdot 6}{81} = \frac{51}{81}.$$

The condition expressed in the hypothesis of Theorem 5 is a necessary one in order to have the formula given before.

Example 4. *Keeping the same group S_3 and considering a different partition of the complete hypergroup H of cardinality 9, as for example $A_e = \{a_0, a_1\}$, $A_{(12)} = \{a_2\}$, $A_{(23)} = \{a_3\}$, $A_{(31)} = \{a_4, a_5\}$, $A_{(123)} = \{a_6\}$, and $A_{(321)} = \{a_7, a_8\}$, then we find that the elements a_2 and a_5 have the same conjugacy classes, i.e., $[a_2] = A_{(12)} \cup A_{(23)} \cup A_{(31)} = \{a_2, a_3, a_4, a_5\} = [a_5]$, while they have different centralizers: $C_H(a_2) = A_e \cup A_{(12)} = \{a_0, a_1, a_2\}$ and $C_H(a_5) = A_e \cup A_{(31)} = \{a_0, a_1, a_4, a_5\}$. Thus $|C_H(a_2)| \neq |C_H(a_5)|$, so we cannot apply the formula found in Theorem 5, but just the definition of the commutativity degree.*

2.4. The Euler's Totient Function

We start again with the basic concept related to this aspect in group theory, where by $o(a)$ we denote the order of the element a in the group G and $exp(G)$ is the exponent of the group. Then the Euler's totient function has the form:

$$\varphi(G) = |\{a \in G \mid o(a) = exp(G)\}|.$$

Since in a hypergroup we may have or not identities, the role of the order of an element (as in group theory) is taken by the concept of the period of an element, defined as $p(a) = min\{k \in \mathbb{N} \mid a^k \subseteq \omega_H\}$ and then we define [15] the Euler's totient function as

$$\varphi(H) = |\{a \in H \mid p(a) = exp(H)\}|.$$

In particular, in a complete hypergroup $H = \bigcup_{g \in G} A_g$ with the underlying group G, the period of the element $a \in A_g$ is the same as the order of g in G and then $exp(G) = exp(H)$ and the Euler's totient function has the form

$$\varphi(H) = |\{x \in \bigcup_{g \in G} A_g \mid x \in A_g, o(G) = exp(G)\}| = \sum_{o(g)=exp(G)} |A_g|.$$

Example 5. *Continuing with Example 1, we get $\omega_H = A_e = \{a_0\}$. Calculating the periods of all elements in H, we find $p(a_0) = 1$, $p(a_i) = 2$, for $1 \leq i \leq 6$, and $p(a_7) = p(a_8) = 3$. Thus, $exp(H) = 3 = exp(S_3)$ and thereby $\varphi(H) = |\{x \in H \mid p(x) = exp(H)\}| = 2$.*

Calculating the orders of the elements of the group S_3, we get $\varphi(H) = \sum_{o(g)=exp(G)=3} |A_g| = |A_{(123)}| + |A_{(321)}| = 1 + 1 = 2$, so the formula is verified.

3. Conclusions

A similar study investigating these main properties related to the above mentioned arithmetic functions can be conducted also on other types of hypergroups, as for example cyclic hypergroups, canonical hypergroups or HX-groups, emphasyzing the differences with respect to groups.

Funding: This research received no external funding.

References

1. Jun, J. Algebraic geometry over hyperrings. *Adv. Math.* **2018**, *323*, 142–192.
2. Connes, A.; Consani, C. The hyperring of adele classes. *J. Number Theory* **2011**, *131*, 159–194.

3. Křehlik, Š.; Novak, M.; Vyroubalova, J. From Automata to Multiautomata via Theory of Hypercompositional Structures. *Mathematics* **2021**, *10*, 1.
4. Kalampakas, A.; Spartalis, S.; Tsigkas, A. The Path Hyperoperation. *An. St. Univ. "Ovidius" Constanta Ser. Mat.* **2014**, *22*, 141–153.
5. Baker, M.; Bowler, N. Matroids over partial hyperstructures. *Adv. Math.* **2019**, *343*, 821–863.
6. Jun, J. Association schemes and hypergroups. *Commun. Alg.* **2018**, *46*, 942–960.
7. Koskas, M. Groupoides, demi-hypergroupes et hypergroupes. *J. Math. Pure Appl.* **1970**, *49*, 155–192.
8. Corsini, P. *Prolegomena of Hypergroup Theory*; Aviani Editore: Tricesimo, Italy, 1993.
9. Corsini, P.; Leoreanu, V. *Applications of Hyperstructures Theory*; Kluwer Academic Publishers: Dodrecht, The Netherlands; Boston, MA, USA; London, UK, 2003.
10. Massouros, C.; Massouros, G. An overview of the foundations of the hypergroup theory. *Mathematics* **2021**, *9*, 1014.
11. Massouros, G.; Massouros, C. Hypercompositional algebra, computer science and geometry. *Mathematics* **2020**, *8*, 1338.
12. Jantosciak, J. Reduced hypergroups. In Proceedings of the Algebraic Hyperstructures and Applications: Proceedings of 4th International Congress, Xanthi, Greece, 27–30 June 1990; World Scientific: Singapore, 1991; pp. 119–122.
13. Kankaras, M.; Cristea, I. Fuzzy reduced hypergroups. *Mathematics* **2020**, *8*, 263.
14. Sonea, A.; Cristea, I. The class equation and the commutativity degree for complete hypergroups. *Mathematics* **2020**, *8*, 2253.
15. Sonea, A.; Cristea, I. Euler's totient function applied to complete hypergroups. *AIMS Math.* **2023**, *8*, 7731–7746.