AN ACCELERATED ITERATIVE TECHNIQUE: THIRD REFINEMENT OF GAUSS SEIDEL ALGORITHM FOR LINEAR SYSTEMS

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Introduction

Solving a large linear system is one of the challenges of most modeling problems today. A linear system can be expressed in the format:

$$At = b \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$ is a matrix of coefficients, $b \in \mathbb{R}^n$ is a column of constants and t is an unknown vector to be determined. The partitioning of A gives

$$A = D - G - H \tag{2}$$

In which D is the diagonal component and $-H$, $-G$ are the strictly upper and bottom triangular constituents of A respectively.

Introduction

Statement Problem

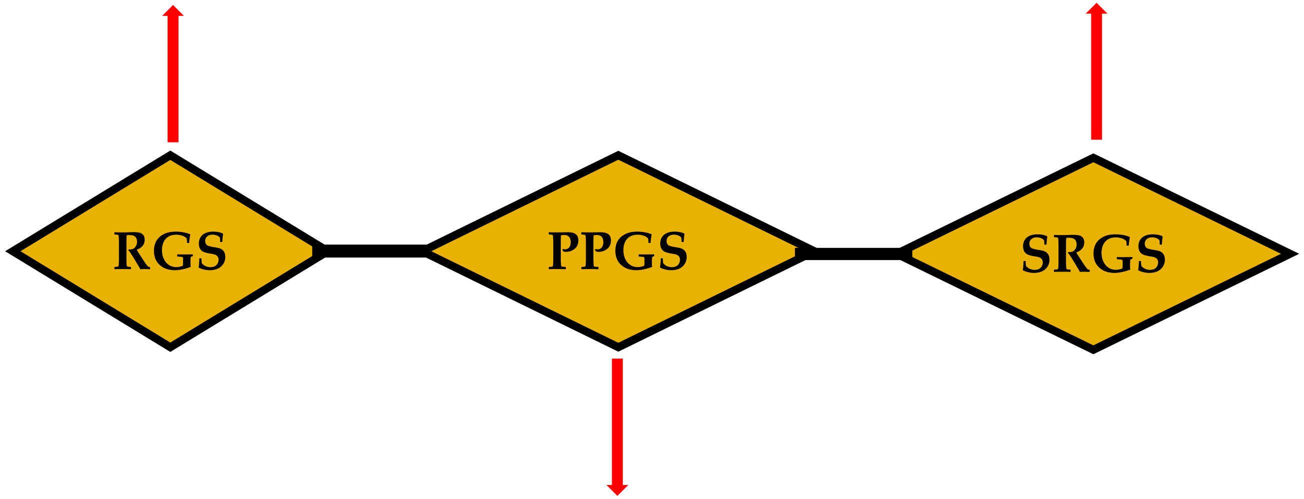
- Iterative approaches are unquestionably the most effective approach to employ, when solving linear systems.
- However, such approach may require several rounds to converge, which reduces computer storage and computing performance.
- In such cases, it is vital to modify or redesign existing methods in order to achieve approximate solutions with rapid convergence.
- This motivated the current study to offer an accelerated technique capable of providing better solutions quickly.

Introduction

Related Research

(2011) Vatti and Kebede
Introduced the method of
Refinement Gauss-Seidel

(2020) Kebede et al.
Introduced the method of Second-
Refinement Gauss-Seidel



(2016) Kumar et al.
Introduced the method of
Parametric Preconditioned
Gauss-Seidel

Introduction

Current Research

- In this study, a third refinement of Gauss-Seidel method for solving linear systems is proposed.

- The convergence properties of the technique were examined.

- The proposed technique was employed to solve linear systems.

Methodology

Considering a linear system of (1), combination of (1) and (2) and process gives the Gauss-Seidel method as

$$t^{(n+1)} = (D-G)^{-1} Ht^{(n)} + (D-G)^{-1} b \qquad (3)$$

The general format of refinement approach is

$$t^{(n+1)} = \tilde{t}^{(n+1)} + (D-G)^{-1}\left(b - A\tilde{t}^{(n+1)}\right) \qquad (4)$$

Then Refinement of Gauss-Seidel is obtained as

$$t^{(n+1)} = \left[(D-G)^{-1} H\right]^2 t^{(n)} + \left[I + (D-G)^{-1} H\right](D-G)^{-1} b \qquad (5)$$

Methodology

Modification of (5) results into

$$t^{(n+1)} = \left[(D-G)^{-1} H\right]^3 t^{(n)} + \left[I + (D-G)^{-1} H + \left((D-G)^{-1} H\right)^2\right](D-G)^{-1} b \qquad (6)$$

We remodel (6) to obtain

$$t^{(n+1)} = \left[(D-G)^{-1} H\right]^4 t^{(n)} + \left[I + (D-G)^{-1} H + \left((D-G)^{-1} H\right)^2 + \left((D-G)^{-1} H\right)^3\right](D-G)^{-1} b \qquad (7)$$

Equation (7) is called Third Refinement of Gauss-Seidel (TRGS) method.

Convergence Analysis

The third refinement of Gauss-Seidel (TRGS) method converges if the spectral radius of its iteration matrix is less than 1, expressed as;

$$\left[\rho\left((D-G)^{-1}H\right)\right]^{4}<1$$

Also, the closer the spectral radius is to zero, the faster the convergence

Theorem 1: If A is strictly diagonally dominant (SDD) matrix, then the TRGS method converges for any choice of the initial guess $t^{(0)}$.

Theorem 2: If A is an M-matrix, then the TRGS method converges for any initial guess $t^{(0)}$.

Result and Discussion

Applied problem [4]: Consider the linear system of equations;

$$\begin{pmatrix} 4.2 & 0 & -1 & -1 & 0 & 0 & -1 & -1 \\ -1 & 4.2 & 0 & -1 & -1 & 0 & 0 & -1 \\ -1 & -1 & 4.2 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 4.2 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 4.2 & 0 & -1 & -1 \\ -1 & 0 & 0 & -1 & -1 & 4.2 & 0 & -1 \\ -1 & -1 & 0 & 0 & -1 & -1 & 4.2 & 0 \\ 0 & -1 & -1 & 0 & 0 & -1 & -1 & 4.2 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \\ t_8 \end{pmatrix} = \begin{pmatrix} 6.20 \\ 5.40 \\ -9.20 \\ 0.00 \\ 6.20 \\ 1.20 \\ -13.4 \\ 4.20 \end{pmatrix}$$

The true solution of the applied problem is $t = (1.0, 2.0, -1.0, 0.0, 1.0, 1.0, -2.0, 1.0)$

Result and Discussion

Table 1. Comparison of Spectral radius and Convergence rate for the Applied Problem.

Technique	Iteration Step	Spectral Radius	Execution Time (sec)	Convergence Rate
GS	88	0.89530	6.70	0.04803
RGS	44	0.80157	5.53	0.09606
SRGS	30	0.71765	5.00	0.14408
TRGS	22	0.64251	4.10	0.19212

The Table shows that TRGS reduced the number of iteration to one-fourth of GS, half of RGS and a few steps of SRGS. Based on how close their spectral radii are to zero, it is inferred that the TRGS technique has a faster rate of convergence than the initial refinements of GS $\left(\rho(TRGS) < \rho(SRGS) < \rho(RGS) < \rho(GS) < 1\right)$

Result and Discussion

Table 2. Solution of the Applied Problem

Technique	n	$t_1^{(n)}$	$t_2^{(n)}$	$t_3^{(n)}$	$t_4^{(n)}$	$t_5^{(n)}$	$t_6^{(n)}$	$t_7^{(n)}$	$t_8^{(n)}$
GS	1	1.47620	1.63720	-1.44920	0.04476	1.14180	0.91970	-1.95840	0.79746
	2	0.86540	1.96410	-1.02590	-0.02391	0.94982	0.90209	-2.07580	0.94392
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	87	0.99999	2.00000	-1.00000	0.00000	1.00000	1.00000	-2.00000	1.00000
	88	1.00000	2.00000	-1.00000	0.00000	1.00000	1.00000	-2.00000	1.00000
RGS	1	0.86540	1.96410	-1.02590	-0.23917	0.949820	0.90209	-2.07580	0.94392
	2	0.94909	1.94710	-1.05170	-0.04878	0.95370	0.95407	-2.04670	0.95306
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	43	0.99999	2.00000	-1.00000	0.00000	0.99999	0.99999	-2.00000	0.99999
	44	1.00000	2.00000	-1.00000	0.00000	1.00000	1.00000	-2.00000	1.00000
SRGS	1	0.95672	1.95870	-1.05540	-0.06439	0.94007	0.94674	-2.04710	0.95308
	2	0.95952	1.96010	-1.03950	-0.03950	0.96145	0.96206	-2.03740	0.96315
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	29	0.99999	2.00000	-1.00000	0.00000	1.00000	1.00000	-2.00000	1.00000
	30	1.00000	2.00000	-1.00000	0.00000	1.00000	1.00000	-2.00000	1.00000
TRGS	1	0.94909	1.94710	-1.05170	-0.04878	0.95370	0.95407	-2.04670	0.95306
	2	0.96741	1.96790	-1.03170	-0.03170	0.96917	0.96959	-2.03000	0.97042
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	21	0.99999	2.00000	-1.00000	0.00000	0.99999	0.99999	-2.00000	0.99999
	22	1.00000	2.00000	-1.00000	0.00000	1.00000	1.00000	-2.00000	1.00000

Conclusion

- The proposed TRGS method achieve a rapid convergence rate compared to GS, RGS and SRGS methods.

- The technique has a significant improvement in reduction of the number of iteration compared to other initial refinement of GS techniques.

- TRGS produces a qualitative and quantitative shift in solving linear systems.

- The proposed technique presents a much more convenient approach of solving linear systems.

- The proposed technique is more efficient than existing refinements of GS.

References

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Thank You

THANK YOU

SO MUCH

FOR LISTENING.