computer sciences and mathematics forum

MDPI

Proceeding Paper

On Single Server Queue with Batch Arrivals †

Vitaly Sobolev [1,*,‡], Alexander Condratenko [2,‡]

1 Department of Probability Theory and Applied Mathematics, Moscow Technical University of Communications and Informatics, 123423 Moscow, Russia
2 Faculty of Mechanics and Mathematics, Lomonosov Moscow State University, 119991 Moscow, Russia; ae_cond@mech.math.msu.su
* Correspondence: v.n.sobolev@mtuci.ru
† Presented at the 1st International Online Conference on Mathematics and Applications. Available online: https://iocma2023.sciforum.net/.
‡ These authors contributed equally to this work.

Abstract: We consider a queuing system $GI^{\nu}|M|1|\infty$ with arrival of customer batches, general renewal arrivals, exponential service times, single service channel and infinite number of waiting positions, customers are serviced in the order of their arrival. In the stationary case a new form of the probability generating functions of the number of clients in the system is derived. This new form is written in terms of the p.g.f. of the tail distribution function of the number of customers per group and of the p.g.f. of a embedded discrete time homogeneous Markov chain. In a queuing system with batch Poisson arrival flow $M^{\nu}_{\lambda}|M_{\mu}|1|\infty$ the number of customers in the system can be obtained from the normalized tail distribution.

Keywords: queueing system; infinite capacity; server; batch arrivals; renewal process; probability generating functions; embedded Markov chain; distribution of the number of customers

1. Introduction

Many practical application in communication systems, production systems, transportation and stocking systems, information processing systems, etc., can be modelled as queueing system. Therefore the queuing theory is very useful for solving this problems. One of the important types of queueing systems is bulk queuing systems [1]. The batch queues is a class of queues in which arrival or service (or both) is in bulks. Many scientific publications is devoted to this type of queuing systems [2,3].

In this manuscript we consider a batch queueing system $GI^{\nu}|M|1|\infty$. It was considered in the works [4,5]. The brief description of this system is as follows. Customers arrival moments $0 < t_1 < t_2 < \ldots < t_n < \ldots$ constitute a renewal process [6] with the probability generating function $P\{t_n - t_{n-1} < t\} = F(t)$. Customers arrive in batches at a single server queue. At every moment t_n a group of ν_n customers arrives. A collection of this random variables ν_n is independent and identically distributed. Additionally suppose that ν_n are bounded and

$$\alpha(z) = Mz^{\nu_n} = \alpha_1 z + \alpha_2 z^2 + \ldots + \alpha_m z^m \, , \, \alpha_m \neq 0$$

is its generating function. The system has single service channel and service time is exponentially distributed with parameter μ. The queue has infinit capacity and customers are serviced in the order of their arrival.

Let a stochastic process $\xi(t)$ denote the number of customers in the queueing system at time t. The stationary distribution of this process can be described using the probability generating function

$$P(z) = \lim_{t \to \infty} Mz^{\xi(t)} = \sum_{n=0}^{\infty} p_n z^n \, . \tag{1}$$

The probability p_n can be interpreted as the fraction of time that n customers are in the system.

Citation: Sobolev, V.; Condratenko, A. On Single Server Queue with Batch Arrivals. *Comput. Sci. Math. Forum* **2023**, *1*, 0. https://doi.org/

Academic Editor: Firstname Lastname

Published: 28 April 2023

Consider process $\xi(t)$ at arrival moments of batches of customers and denote

$$\xi_n = \xi(t_n - 0),\ n = 1, 2, \dots,\ \xi_1 = 0\,.$$

Then ξ_n describe the number of customers in sistems at the arrival moment of batches of customers t_n. It's obvious [4] that the sequence of ξ_n constitutes a homogenous Markov chain.

The stationary distribution of the chain ξ_n to can be described using the probability generating function

$$\pi(z) = \lim_{n\to\infty} Mz^{\xi_n} = \sum_{k=0}^{\infty} \pi_k z^k\,. \tag{2}$$

We will calculate the stationary distribution of the process $\xi(t)$ by calculating the corresponding distribution in the embedded Markov chain ξ_n.

It is known [4,5] that Markov chain ξ_n has a stationary distribution if and only if

$$\nu = \sum_{k=0}^{m} k\alpha_k < \mu T\,, \tag{3}$$

where $T = \int_0^\infty t dF(t)$ is the average inter-arrival time, and

$$\nu = \mathsf{M}\nu_n = \alpha'(z)\big|_{z=1}$$

is the average number of customers in an arriving batch. The steady state condition (3) of queue can be written in the traffic rate form

$$\rho = \frac{\nu}{\mu T} < 1 \tag{4}$$

where ρ is the traffic rate, here generalized for batch systems. Next, we suppose that the inequality (4) holds.

2. Results

As we will see below, some normalized tail probabilities [7], and only they connect the stationary distribution of the stochastic process $\xi(t)$ with the stationary distribution of the chain ξ_n. Therefore, it will be convenient to use a notation for the distribution tails of ν_n. So we shall write

$$A_k = \mathsf{P}\{\nu_n \geqslant k\} = \sum_{l=k}^{m} \alpha_l\,,\ k = 1, \dots, m$$

and

$$A(z) = \frac{1}{\nu} \sum_{k=1}^{m} A_k z^k\,. \tag{5}$$

In this case, it is easy to see that $A(z)$ is the probability generating function for some discrete random variable ζ with the probability mass function

$$q_k = \mathsf{P}\{\zeta = k\} = A_k/\nu\,,\ k = 1, \dots, m\,.$$

Let 's call the probability distributions given by $\{q_k\}$ the normalized distribution tails and $A(z)$ the probability generating function of the normalized distribution tails.

For $A(z)$ it can be shown that

$$A(z) = \frac{z}{\nu} \frac{1 - \alpha(z)}{1 - z}\,.$$

So, we have the following chain of equalities:

$$\frac{\nu}{z}A(z) = \sum_{l=k}^{m} A_k z^{k-1} =$$

$$= \alpha_1 + \alpha_2(1+z) + \cdots + \alpha_m\left(1+z+\cdots+z^{m-1}\right) =$$

$$= \frac{\alpha_1(1-z) + \alpha_2\left(1-z^2\right) + \cdots + \alpha_m(1-z^m)}{1-z} =$$

$$= \frac{1-\alpha_1 z - \alpha_2 z^2 - \cdots - \alpha_m z^m}{1-z} = \frac{1-\alpha(z)}{1-z}\,.$$

This approach allows us to formulate two theorem.

Theorem 1. *Under the condition stationary distribution of the process $\xi(t)$ exists and it may be defined by the generating funtion*

$$P(z) = \rho\pi(z)A(z) + 1 - \rho\,, \tag{6}$$

where $A(z)$ is the probability generating function (5)*, ρ is traffic rate* (4)*, $\pi(z)$ is defined by* (2)*.*

Remark 1. *Formula* (6) *show that the distribution of the probability p_n is a mixture the degenerate distribution and distribution can be represented as convolution of two distributions: one of which is the distribution of the nested Markov chain, and the other is the normalized tail distribution of the arriving batch sizes.*

Now let us consider a queuing system $M_\lambda^\nu|M_\mu|1|\infty$ with batch Poisson arrival flow in which the arrival rate constant λ, i.e., batch of customers arrive with exponential interarrival times with mean $T = \frac{1}{\lambda}$. In this case , the following theorem holds.

Theorem 2. *For sistem $M_\lambda^\nu|M_\mu|1|\infty$ under* (4) *the condition stationary distribution of the process $\xi(t)$ exists and it may be defined by the generating funtion*

$$P(z) = \pi(z) = \frac{1-\rho}{1-\rho A(z)} \tag{7}$$

where $A(z)$ from (5)*, and ρ is traffic rate* (4)*, $\pi(z)$ is defined by* (2)*.*

3. Conclusions

In this article, we studied relationships among probability distributions in a single server batch queueing model $GI_\lambda^\nu|M_\mu|1|\infty$. Future research may be devoted to the search for similar probabilistic relationships between the target sequence of probabilities and the corresponding nested Markov chain in other queuing systems.

Author Contributions: The authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We wish to express our gratitude to V.V. Kozlov, who was abundantly helpful and offered invaluable assistance for technical support. The authors would like to thank the editors and the referees for their valuable comments and suggestions which have helped in improving the quality and presentation of the paper.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

p.g.f. probability generating functions

References

1. Chaudhry, M.L.; Templeton, J.G.C. *A First Course in Bulk Queues*, 1rd ed.; John Wiley & Sons: Hoboken, NJ, USA, 1983.
2. Daw A., Fralix B., Pender J. Non-Stationary Queues with Batch Arrivals. *arXiv* **2021**, arXiv:1907.12650v3.
3. Niranjan, S.P.; Indhira, K. A review on classical bulk arrival and batch service queueing model. *Int. J. Pure Appl. Math.* **2016**, *106*, 45–51.
4. Soloviev, A.D.; Sobolev, V.N. One Server Queue with Bulk Arrivals. In *Analytical and Computational Methods in Probability Theory, ACMPT 2017*; Rykov, V., Singpurwalla, N., Zubkov, A., Eds.; Springer: Cham, Switzerland, 2007; pp. 97–108.
5. Soloviev, A.D.; Sobolev, V.N. One server queue with bulk arrivals. In Proceedings of the Analytical and Computational Methods in Probability Theory and its Applications (ACMPT-2017), Proceedings of the International Scientific Conference, Moscow, Russia, 23–27 October 2017; Lebedev, A.V., Ed.; RUDN: Moscow, Russia, 2017.
6. Gnedenko, B.V.; Kovalenko, I.N. *Introduction to Queueing Theory*, 2rd ed.; Birkhauser: Boston, NJ, USA, 1989; pp. 108–119.
7. Feller, W. *An Introduction to Probability Theory and Its Applications*, 3rd ed.; John Wiley & Sons, Inc.: New York, NY, USA, 1968; Volume I., pp. 264–266.