A first-order evolution problem with maximal monotone operators

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Contents

In the current work, we aim to study a differential inclusion governed by time and state dependent maximal monotone operators with perturbation. We show that this evolution problem admits bounded variation continuous solution. We provide example that illustrate the established result. has been considered by many authors, see eg[Charles Castaing and Saïdi., 2021, D. Azzam-Laouir and Marques, 2019, D. Azzam-Laouir, 2022, Charles Castaing, 2021]

Notation

H is real separable Hilbert space;
$L^2_H(I)$ is the space of square integrable functions from I to H;
$W^{1,2}(I,H)$ is the space of absolutely continuous functions from I to H with derivatives in $L^2_H(I)$;
$\mathcal{C}_H(I)$ is the space of continuous maps x from I to H;
$A^0(t,x)$ denotes the element of minimal norm of $A(t,x)$;

$$\mathrm{D}\,(A) = \{x \in H : \; Ax \neq \emptyset\},$$

the resolvent of A define by $J^A_\lambda = (I_H + \lambda A)^{-1}$ and the

Let A, B be two maximal monotone operators such that $B : \mathrm{D}(B) \subset H \rightrightarrows H$ and $A : \mathrm{D}(A) \subset H \rightrightarrows H$ and we define the pseudo-distance between A and B by

$$\operatorname{dis}(A, B) = \sup \left\{ \frac{\langle y_1 - y_2, x_2 - x_1 \rangle}{1 + ||y_1|| + ||y_2||} : \ (x_1, y_1) \in Gr(A), \ (x_2, y_2) \in Gr(B) \tag{1}$$

Recall that,in the sense of convex analysis, the normal cone to the set $C(t,x)$ at $v \in H$, for all $(t,x) \in I \times H$ is the subdifferential of the indicator function of the set at v that is

$$N_C(t,x) = \partial\delta_C(t,x) = \{\zeta \in H; \langle \zeta, z - y \rangle \leq 0 \forall z \in C(t,x)\}$$

which is maximal monotone operator with $\mathrm{D}\left(N_{C(t,x)}\right) = C(t,x)$ Furthermore, we know that for all $t, s \in I$ and $x, y \in H$,

$$\mathrm{dis}\left(N_{C(t,x)}, N_{C(s,y)}\right) = \mathcal{H}(C(t,x), C(s,y))$$

Lemma

Let B_n $(n \in \mathbb{N})$, B be maximal monotone operators of H such that $dis(B_n, B) \to 0$. Suppose also that $x_n \in D(B_n)$ with $x_n \to x$ and $y_n \in B_n(x_n)$ with $y_n \to y$ weakly for some $x, y \in H$. Then $x \in D(B)$ and $y \in B(x)$.

Lemma

Let B_n $(n \in \mathbb{N})$, B be maximal monotone operators of H such that $dis(B_n, B) \to 0$ and $\|B_n^0(x)\| \leq c(1 + \|x\|)$ for some $c > 0$, all $n \in \mathbb{N}$ and $x \in D(B_n)$. Then for every $z \in D(B)$ there exists a sequence (z_n) such that

$$z_n \in D(B_n), \quad z_n \to z \text{ and } B_n^0(z_n) \to B^0(z).$$

Assume that for any $(t, y) \in I \times H$, $B(t, y) : D(B(t, y)) \subset H \rightrightarrows H$ is a maximal monotone operator satisfying

(H_B^1) There exist a non-negative real constant $\lambda < \frac{2}{3}$, and a function $r : [0, T] \to [0, +\infty[$ which is continuous on $[0, T]$ and non-decreasing with $r(T) < \infty$ such that

$$dis(B(t, y), B(s, z)) \leq r(t) - r(s) + \lambda \|y - z\|, \text{ for } 0 \leq s \leq t \leq T, \text{ for } y, z \in$$

(H_B^2) There exists a non-negative real number c such that

$$\|B^0(t, y)z\| \leq c(1 + \|y\| + \|z\|) \text{ for } t \in I,\ y \in H,\ z \in D(B(t, y)).$$

(H_B^3) For any bounded subset F of H, the set $D(B(I \times F))$ is relatively ball-compact.

Let $f : I \times H \to H$ be a map such that

(H_f') for any fixed $x \in H$, $f(\cdot, x)$ is measurable on I and for any fixed $t \in I$, $f(t, \cdot)$ is continuous on H;

(H_f'') there exists a non-negative real constant L such that

$$f(t, x) \leq L(1 + ||x||) \text{ for all } (t, x) \in I \times H. \qquad (2)$$

Main result

Then, for any $(u_0) \in D(B(0, u_0)) \times H$, the evolution problem

$$\begin{cases} -\dfrac{du}{dr}(t) \in B(t, u(t))u(t) + f(t, u(t)) & \mathrm{dr} - \text{a.e. } t \in I := [0, T] \\ u(t) \in D(B(t, u(t))), & t \in I \end{cases} \tag{3}$$

admits a BV continuous solution $u : I \to H$
Furthermore, $u'(t) = \frac{du}{dr}(t)$ with respect to dr, and

$$||u'(t)|| \leq k_3 \;\; dr - \text{a.e. } t \in I,$$

and

$$\|u(t) - u(s)\| \leq k_3 \Big(r(t) - r(s) \Big) \text{ for } 0 \leq s \leq t \leq T,$$

for a real positive constant k_3 depending on c, T, λ, $r(T)$.

proof

(I) Let us construct the sequences (u_n) .
For any $n \geq 1$, define a partition of $I := [0, T]$ with

$$0 = t_0^n < t_1^n < \cdots < t_i^n < t_{i+1}^n < \cdots < t_n^n = T.$$

For any $n \geq 1$ and $i = 0, 1, \cdots, n-1$, set

$$r_{i+1}^n = r(t_{i+1}^n) - r(t_i^n), \quad , r_i^n \leq r_{i+1}^n \leq k_n = \frac{r(T)}{n}, \qquad (4)$$

fix any $n \geq 1$. Put $u_0^n = u_0 \in D(B(0, u_0))$. For $i \in \{0, \cdots, n-1\}$ choose and set

$$u_{i+1}^n = J_{i+1}^n \left(u_i^n - \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n) dr(s) \right), \tag{5}$$

one writes

$$-\frac{1}{r_{i+1}^n} \left(u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n) dr(s) \right) \in B(t_{i+1}^n, u_i^n) u_{i+1}^n. \tag{6}$$

Now by the fact that $(ef)^{\frac{1}{2}} \leq \frac{1}{2}e + \frac{1}{2}f$ for positive real constants e, f, yields then by assumption before one gets

$$\|u_{i+1}^n - u_i^n\| \leq r_{i+1}^n(L + \frac{3c}{2} + 2)(1 + \|u_i^n\| + \|u_{i-1}^n\|) + \frac{3c}{2}\|u_i^n - u_{i-1}^n\|.$$

then Remember that $\lambda < \frac{2}{3}$, then setting $\mu = \frac{3\lambda}{2}$ and $\eta = (L + \frac{3c}{2} + 2)$, by iteration

$$||u_{i+1}^n - u_i^n|| \leq r_{i+1}^n \eta \sum_{j=0}^{i} \mu^j (1 + ||x_{i-j}^n|| + ||u_{i-j-1}^n||). \quad (7)$$

Thus, for any n and $i = 0, \cdots, n-1$

$$||u_{i+1}^n|| \leq ||u_0^n|| + \sum_{j=0}^{i} ||u_{j+1}^n - u_j^n|| \tag{8}$$

$$\leq \left(||x_0|| + \eta \frac{r(T)}{1-\mu} \right) \exp\left(\frac{3\eta r(T)}{1-\mu}\right) = k_1 \tag{9}$$

This,along with(7) yields

$$||u_{i+1}^n - u_i^n|| \leq \frac{\eta(1+2M_1)}{1-\mu} r_{i+1}^n = k_2 r_{i+1}^n. \tag{10}$$

For any $n \geq 1$, define the following sequences u_n for all $t \in [t_i^n, t_{i+1}^n[$, $i \in \{0, \cdots, n-1\}$ by

$$u_n(t) = u_i^n + \frac{r(t) - r(t_i^n)}{r(t_{i+1}^n) - r(t_i^n)} \left(u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n(s)) dr(s) \right) - \int_{t_i^n}^{t} f(s, u_i^n) dr(s), \quad (11)$$

and Put for any $n \geq 1$

$$\delta_n(t) = \begin{cases} t_i^n & \text{if } t \in]t_i^n, t_{i+1}^n] \quad \text{for some } i \in \{0, 1, \cdots, n-1\}, \\ 0 & \text{if } t = 0, \end{cases}$$

$$\phi_n(t) = \begin{cases} 0 & \text{if } t = 0, \\ t_{i+1}^n & \text{if } t \in]t_i^n, t_{i+1}^n] \quad \text{for some } i \in \{0, 1, \cdots, n-1\}. \end{cases}$$

by derivation one gets for all $t \in]t_i^n, t_{i+1}^n[$

$$\frac{du_n}{dr}(t) = \frac{1}{r_{i+1}^n} \left(u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n) dr(s) \right) - f(s, u_i^n), \quad (12)$$

Hence, by (6)for each $n \in \mathbb{N}^*$, there is a null Lebesgue measure set $K_n \subset I$ such that

$$-\frac{du_n}{dr}(t) \in B(\delta_n(t), u_n(\delta_n(t)))u_n(\phi_n(t)) + f(t, u_i^n)\,\mathrm{dr} - \mathrm{a.e}\ t \in I, \tag{13}$$

$$u_n(\delta_n(t)) \in D(B(\delta_n(t), u_n(\delta_n(t)))),\ t \in I. \tag{14}$$

In view of (4), (10) and (12) one has for all $t \in [t_i^n, t_{i+1}^n[$

$$\|\frac{du_n}{dr}(t)\| \leq 2k_2 + 2L(1 + k_1) = k_3$$

Put $f_n(t) = f(t, u_n(\delta_n(t)))$

Thanks to (14) and , one gets for all $t \in I$

$$(u_n(\delta_n(t))) \subset D(B(I \times k_1\overline{B}_H)) \cap k_1\overline{B}_H. \tag{15}$$

These inclusions along with (H_B^3) entail that the set $\{u_n(\delta_n(t)) : n \in \mathbb{N}^*\}$ is relatively compact in H.
the absolute continuity of u_n for any n, one has

$$\|u_n(\delta_n(t)) - u_n(t)\| = \left\| \int_t^{\delta_n(t)} \frac{du}{dr} dr \right\| \leq |r(\delta_n(t)) - r(t)|^{\frac{1}{2}} k_3.$$

Hence, the set $\{u_n(t) : n \in \mathbb{N}^*\}$ is relatively compact in H.
for any $s, t \in I$, $t \leq s$

$$\|u_n(s) - u_n(t)\| = \left\| \int_t^s \frac{du}{dr} d\tau \right\| \leq \int_t^s \|\frac{du}{dr}\| d\tau \leq \|s-t\|^{\frac{1}{2}} k_3, \tag{16}$$

that is, $\{u_n(\cdot) : n \in \mathbb{N}^*\}$ is equicontinuous. By Ascoli's theorem, $(u_n(\cdot))_n$ is relatively compact in $\mathcal{C}_H(I)$. then

$$\|u_n(\delta_n(t)) - u(t)\| \to 0, as \quad n \to \infty. \tag{17}$$

Now we have that u_n continue and converge to $u(t)$ then

$$\lim_{n\to\infty}\langle e, u_n(t) - u_n(s)\rangle = \lim_{n\to\infty}\langle e, \int_s^t \frac{du_n}{dr}(\tau)d\tau\rangle \tag{18}$$

(19)

$$\langle e, u(t) - u(s)\rangle = \left\langle e, \int_s^t z(\tau)d\tau \right\rangle.$$

Hence, given any $s, t \in [0, T]$ with $s \leq t$, we get $\int_s^t z(\tau)d\tau = u(t) - u(s)$, and then $u(\cdot)$ is absolutely continuous and $z(\cdot)$ coincides almost everywhere in $[0, T]$ with $\frac{du}{dr}(\cdot)$. Moreover, it results

$$\frac{du_n}{dr} \to \frac{du}{dr} \text{ weakly in } L^2(I, H, dr). \tag{20}$$

We show

$$u(t) \in D(B(t, u(t))), \quad t \in I, \tag{21}$$

$$u(0) = u_0 \in D(B(0, u_0)),\ u(0) = u_0 \in H.$$

Recall that $u_n(\delta_n(t)) \in D(B(\delta_n(t), u_n(\delta_n(t))))$ for all $t \in I$ (see (14)). Combining (H_B^1), (4), (4) yields

$$dis(B(\delta_n(t), u_n(\delta_n(t))), B(t, u(t))) \to 0 \text{ as } n \to \infty. \tag{22}$$

Remark that in view of (H_B^2), (17), $(w_n) = (B^0(\delta_n(t), u_n(\delta_n(t)))u_n(\delta_n(t)))$ is bounded. Then, we may extract from (w_n) a subsequence that weakly converges to $w \in H$. Since the sequence $(u_n(\delta_n(t)))$ converges to $u(t)$ in H applying Lemma 1, one concludes that $u(t) \in D(B(t, u(t)))$, $t \in I$.

now let proof

$$-\frac{du}{dr}(t) \in B(t, u(t))u(t) + f(t, u(t)) \quad \mathrm{dr-a.e.}\ t \in I, \qquad (23)$$

it suffices to show that

$$\langle \frac{du}{dr}(t) + f(t, u(t)), u(t) - z\rangle \le \langle B^0(t, u(t))z, z - u(t)\rangle \quad \mathrm{dr-a.e.}\ t \in I,$$

we know that $(\frac{du_n}{dr} + f_n(\cdot))$ weakly converges to $\frac{du}{dr}(\cdot) + f(\cdot, u(\cdot))$ in $L^2(I, H, dr)$. Then, there exists a sequence (ζ_j) such that for each $j \in \mathbb{N}$, $\zeta_j \in co\{\frac{du_k}{dr} + f_k(t),\ k \ge j\}$ and (ζ_j) strongly converges to $\frac{du}{dr}(\cdot) + f(\cdot)$ in $L^2(I, H, dr)$. In other words, there exists a subset K of I with null-Lebesgue measure and a subsequence (j_p) of $\mathbb{N}$ such that for all $t \in I \setminus K$, $(\zeta_{j_p}(t))$ converges to $\frac{du}{dr}(t) + f(s, u(t))$. Hence, for $t \in I \setminus K$

$$\frac{du}{dr}(t) + f(t, u(t)) \in \bigcap_{p\in\mathbb{N}} \overline{co}\{\frac{du_k}{dr}(t) + f_n(t),\ k \ge j_p\},$$

imply that for $t \in I \setminus K$ and any $w \in H$

$$\langle \frac{du}{dr}(t) + f(t), w\rangle \le \limsup_{n\to\infty} \langle \frac{du_n}{dr}(t) + f_n(t), w\rangle. \qquad (24)$$

apply Lemma 2 to $B(\delta_n(t), u_n(\delta_n(t)))$ and $B(t, u(t))$ to ensure the existence of a sequence (z_n) such that $z_n \in D(B(\delta_n(t), u_n(\delta_n(t))))$

$$z_n \to z \text{ and } B^0(\delta_n(t), u_n(\delta_n(t)))z_n \to B^0(t, u(t))z. \qquad (25)$$

For $n \geq 1$, let $I \setminus K_n$ denote the set on which (13) holds. Since $B(t, y)$ is monotone for any $(t, y) \in I \times H$, one obtains for $t \in I \setminus K_n$

$$\langle \frac{du_n}{dr}(t) + f_n(t), u_n(\delta_n(t)) - z_n \rangle \leq \langle B^0(\delta_n(t), u_n(\delta_n(t)))z_n, z_n - u_n(\delta_n(t)) \rangle. \qquad (26)$$

then

$$\langle \quad \frac{du_n}{dr}(t) + y_n(t), u(t) - z\rangle \leq \langle B^0(\delta_n(t), u_n(\delta_n(t)))z_n, z_n - u_n(\delta_n(t))\rangle$$
$$+\Big(k_1 + L(1+k_1)\Big)\Big(\|u_n(\delta_n(t)) - u(t)\| + \|z_n - z\|\Big).$$

which give

$$\limsup_{n\to\infty}\langle\frac{du_n}{dr}(t) + f_n(t), u(t) - z\rangle \leq \langle B^0(t, x(t))z, z - u(t)\rangle.$$

The inclusion

$$-\frac{du}{dr}(t) \in B(t, u(t))u(t) + f(s, u(t)) \quad \mathrm{dr-a.e.}\ t \in I, \qquad (27)$$

therefore holds true.

Let $C : I \times H \rightrightarrows H$ be a set-valued mapping satisfying:
(H_1') For each $t \in I$, $C(t, y)$ is a non-empty closed convex subset of H.
(H_2') There exist a non-negative real constant $\lambda < \frac{2}{3}$, and a function $r : [0, T[\to [0, \infty[$ which is continue on $[0, T[$ and non-decreasing with $r(T) < \infty$ and $r(0) = 0$ such that

$$|d(x, C(t, u)) - d(x, C(s, v))| \leq |r(t) - r(s)| + \lambda ||v - u|| \ \forall t, s \in I, \ \ \forall x, v, u$$

here $\mathcal{H}$ stands for the Hausdorff distance between closed subsets of H
Let $C : I \times H \rightrightarrows H$ be a set-valued mapping satisfying:
(H_1'), (H_2') Let $f : I \times H \to H$ be a map such that (H_f'), (H_f'') then for any $u_0 \in C(0, u_0)$ the problem

$$\begin{cases} -\dfrac{du}{dr}(t) \in N_{C(t,u(t))}u(t) + f(t, u(t)) \quad \mathrm{dr\,a.e.}\ t \in I, \\ u(t) \in C(t, x(t)), \quad t \in I \\ u(0) = u_0 \in C(0, x_0),\ x(0) = x_0 \in H; \end{cases}$$

has an bounded variation continuous solution $u : I \to H$.

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Thanks for your attention