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Tame Topology †

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Abstract: Alexander Grothendieck suggested creating a new branch of topology, called by him "topologie modérée". In the paper "On Grothendieck's tame topology" by N. A'Campo, L. Ji, and A. Papadopoulos (Handbook of Teichmüller Theory, Volume VI. IRMA Lectures in Mathematics and Theoretical Physics Vol. 27 (2016), pp. 521–533) the authors conclude that no such tame topology has been developed on the purely topological level. We see our theory of sets with distinguished families of subsets, which we call smopologies, as realising Grothendieck's idea and the demands of the mentioned paper. Dropping the requirement of stability under infinite unions makes getting several equivalences of categories of spaces with categories of lattices possible. We show several variants of Stone Duality and Esakia Duality for categories of small or locally small spaces and some subclasses of strictly continuous (or bouned continuous) mappings. Such equivalences are better than the spectral reflector functor for usual topological spaces. In particular, spectralifications of Kolmogorov locally small spaces can be obtained by Stone Duality. Small spaces or locally small spaces seem to be generalised topological spaces. However, looking at them as topological spaces with additional structure is better. The language of smopologies and bounded continuous mappings simplifies the language of certain Grothendieck sites and permits us to glue together infinite families of definable sets in structures with topologies, which was important in the case of developing o-minimal homotopy theory.

Keywords: tame topology; Stone Duality; Esakia Duality; spectralification; Grothendieck site; o-minimal structure; equivalence of categories

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1. Introduction

1.1. Grothendieck's Programme and Basics of Tame Topology

Alexander Grothendieck suggested in his famous scientific programme [1] creating a new kind of topology, called by him "topologie modérée" ("tame topology" in English) that would eliminate pathological phenomena (for example, space-filling curves). His mathematical ideas led to philosophical and physical questions about the nature and structure of space ([2]). Grothendieck's programme was realised in many special situations for many decades, but N. A'Campo, L. Ji and A. Papadopoulos [3] state that no clear definition of tame topology has been given. One could say that, as a part of model theory or real algebraic geometry, we have o-minimality (the main reference is [4]), which is widely recognised as a realisation of Grothendieck's programme. But in o-minimality, the definable open sets, not arbitrary open sets, play the main role. This means that, from the tame point of view, the usual notion of a topology is secondary to another concept basic to some algebra-friendly topology. We propose the theory of *tame spaces* (such as *small spaces* and *locally small spaces*) as a realisation of Grothendieck's postulate on a purely topological level. Seemingly a kind of generalised topology, but in fact, tame topology is the usual topology with some additional structure.

Definition 1 ([5–7])**.** *A* locally small space *is a pair* $\mathcal{X} = (X, \mathcal{L}_X)$*, where* X *is any set and* $\mathcal{L}_X \subseteq \mathcal{P}(X)$ *satisfies the following conditions:*

(LS1) $\emptyset \in \mathcal{L}_X$,
(LS2) *if* $A, B \in \mathcal{L}_X$*, then* $A \cap B, A \cup B \in \mathcal{L}_X$,
(LS3) $\forall x \in X \, \exists A_x \in \mathcal{L}_X \; x \in A_x$ *(i.e.,* $\bigcup \mathcal{L}_X = X$*).*

Elements of $\mathcal{L}_X$ *are called* small open *subsets (or* smops*) of* X*, while* $\mathcal{L}_X$ *is called a* smopology*. A* small space *is such a locally small space* $(X, \mathcal{L}_X)$ *that* $X \in \mathcal{L}_X$*. Then the smopology is called* unitary*. The complements of smops are called* co-smops*, and the Boolean combinations of smops are the* constructible sets*. The families of all co-smops (constructible sets, resp.) of a small space* $(X, \mathcal{L}_X)$ *is denoted by* $\mathcal{L}'_X$ *(*$Con(\mathcal{X})$*, resp.).*

As we see, the idea is to drop some of the conditions for a topology. The finitary character of small spaces distinguishes them among locally small spaces. A smopology is a basis of a usual topology. Small spaces were used explicitely in [5–10], while locally small spaces were used explicitely in [5,6,9,10], sometimes with another definition.

Notation. For families of subsets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$, we use

$$\mathcal{A}^o = \{Y \subseteq X : Y \cap A \in \mathcal{A} \text{ for any } A \in \mathcal{A}\},$$

$$\mathcal{A} \cap_1 \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}.$$

1.2. Genealogy and Implicit use of Tame Spaces

Small spaces are pretty often unnamed in the literature ([11], Definition 7.1.14 or [12], p. 12). Both small and locally small spaces were implicitly used in o-minimal homotopy theory ([13,14]) under the name of generalised topological spaces (in the sense of Delfs and Knebusch), which in turn may be seen as sets with G-topologies (compare [15]) or a particular form of Grothendieck sites (see [8], ([14], p. 2)). Definable or locally definable spaces, widely used in real algebraic geometry or model theory ([4,8,14,16], implicitly even [17]), are expansions of small or locally small spaces, respectively.

Definition 1 above gives a simple language for locally small spaces, not using Grothendieck sites, which is analogical to Lugojan's ([18]) or Császár's ([19]) language of generalised topology, where a family of subsets is required to have some, but not all, conditions traditionally required for a topology.

1.3. Categories of Tame Spaces

Having locally small spaces and small spaces, we need to distinguish important mappings between them. That is why we use the following notions.

Definition 2 ([5–7])**.** *Assume* $(X, \mathcal{L}_X)$ *and* $(Y, \mathcal{L}_Y)$ *are locally small spaces. Then a mapping* $f : X \to Y$ *is:*

(a) bounded *if* $\mathcal{L}_X$ *refines* $f^{-1}(\mathcal{L}_Y)$*: each* $A \in \mathcal{L}_X$ *admits* $B \in \mathcal{L}_Y$ *such that* $A \subseteq f^{-1}(B)$,
(b) continuous *if* $f^{-1}(\mathcal{L}_Y) \cap_1 \mathcal{L}_X \subseteq \mathcal{L}_X$ *(i.e.,* $f^{-1}(\mathcal{L}_Y) \subseteq \mathcal{L}^o_X$*),*
(c) strongly continuous *if* $f^{-1}(\mathcal{L}_Y) \subseteq \mathcal{L}_X$,
(d) *a* strict homeomorphism *if* f *is a bijection and* $f^{-1}(\mathcal{L}_Y) = \mathcal{L}_X$.

It is suitable to involve the category theory language. The spaces satisfying the Kolmogorov separation axiom (T_0) are in our focus.

Definition 3 ([5,6])**.** *We have the following categories:*

(a) *the category* **LSS** *of locally small spaces and their bounded continuous mappings,*
(b) *the full subcategory* $\mathbf{LSS}_0$ *of* T_0 *locally small spaces,*
(c) *the full subcategory* $\mathbf{SS}_0$ *of* T_0 *small spaces.*
(d) *the subcategory* $\mathbf{LSS}_0^s$ *in* $\mathbf{LSS}_0$ *of (bounded) strongly continuous mappings.*

Definition 4 ([7]). *The topology $\tau(\mathcal{L}_X)$ generated by $\mathcal{L}_X$ is called the* original topology *and the topology $\tau(Con(\mathcal{X}))$ generated by $Con(\mathcal{X})$ is called the* constructible topology *of $(X, \mathcal{L}_X)$.*

A small space $(X, \mathcal{L}_X)$ is called Heyting *if it is T_0 and the closure in the original topology of any constructible set is a co-smop (i.e., $\overline{A} \in \mathcal{L}'_X$ for any $A \in Con(\mathcal{X})$).*

A map between Heyting small spaces $f\colon \mathcal{X} \to \mathcal{Y}$ is Heyting continuous *if it is continuous and satisfies any of the following equivalent conditions:*

1. $f^{-1}(\overline{C}) = \overline{f^{-1}(C)}$ *for $C \in Con(\mathcal{Y})$,*
2. $f^{-1}(int(C)) = int(f^{-1}(C))$ *for $C \in Con(\mathcal{Y})$.*

We have the category **HSS** *of Heyting small spaces and Heyting continuous maps.*

The name "Heyting small spaces" follows the conventions of [20].

1.4. Stone and Esakia Dualities

Although the language of category theory was not developed in 1930's, Stone Duality has its name after the papers of M. H. Stone ([21,22] for generalised Boolean algebras, Ref. [23] for distributive lattices). There exist plenty of available versions ([20,24,25]), including versions developed by H. Priestley ([26,27]) and named Priestley Duality. Esakia Duality, while emerged from the considerations on modal logics ([28,29]), can be seen as a restriction of Priestley Duality.

Algebraic and analytic geometry as well as model theory use Stone Duality. The spectral topology (also called the Harrison topology) is used in the case of the real spectrum (see [11,12,30]) and the Zariski spectrum (see [31,32], Chapter II), while the constructible topology (also called the patch topology) in the case of the space of types ([33,34]), allowing (in the case of the o-minimal spectrum) retopologisation to the spectral topology ([35]).

There are many extensions of Stone Duality published in recent years. For example: the locally compact Hausdorff case ([36]), removing the zero-dimensionality together with the commutativity assumptions ([37]), a generalisation of Gelfand–Naimark–Stone Duality to completely regular spaces ([38]) and application to the characterisation of normal, Lindelöf, locally compact Hausdorff spaces ([39]), dropping completely the compactness assumption ([40]). From the algebraic side, we have extensions to: orthomodular lattices ([41]), some non-distributive (implicative, residuated, or co-residuated) lattices ([42]), and left-handed skew Boolean algebras ([43]). Esakia Duality has an extension to implicative semilattices ([44]). Many applications of Stone Duality exist in various contexts ([45–47]).

2. Results

Definition 5 ([6]). *A* bornology *in a bounded lattice $(L, \vee, \wedge, 0, 1)$ is an ideal $B \subseteq L$ such that $\bigvee B = 1$. The set of all prime filters in L is denoted by $\mathcal{PF}(L)$. For each $a \in L$, we have $\tilde{a} = \{F \in \mathcal{PF}(L) \mid a \in F\}$. We set $\widetilde{A} = \{\tilde{a} \mid a \in A\} \subseteq \mathcal{P}(\mathcal{PF}(L))$ for $A \subseteq L$.*

2.1. Categories of Distributive Lattices

Definition 6 ([6,7]). *An* object of **LatBD** *is a system $(L, L_s, \mathbf{D}_L)$ with $L = (L, \vee, \wedge, 0, 1)$ a bounded distributive lattice, L_s a bornology in L and $\mathbf{D}_L \subseteq \mathcal{PF}(L)$ (a* decent lump*) satisfying the conditions:*

(1) $\mathbf{D}_L \subseteq \bigcup \widetilde{L_s}$,
(2) $\forall a, b \in L \quad a \neq b \implies \tilde{a}^d \neq \tilde{b}^d$, *where $\tilde{a}^d = \{F \in \mathbf{D}_L \mid a \in F\}$,*
(3) $\widetilde{L} \cap_1 \mathbf{D}_L = (\widetilde{L_s} \cap_1 \mathbf{D}_L)^o \subseteq \mathcal{P}(\mathbf{D}_L)$.

A morphism of **LatBD** *from $(L, L_s, \mathbf{D}_L)$ to $(M, M_s, \mathbf{D}_M)$ is such a homomorphism of bounded lattices $h : L \to M$ that:*

(a) *satisfies the* condition of domination $\forall a \in M_s\ \exists b \in L_s \quad a \vee h(b) = h(b)$,
(b) *respects the decent lump: $\{h^{-1}(G) : G \in \mathbf{D}_M\} \subseteq \mathbf{D}_L$.*

The category **LatD** *may be identified with the full subcategory in* **LatBD** *generated by objects satisfying $L = L_s$.*

Definition 7 ([6,7])**.** *The category* **ZLatD** *has*

(1) *pairs* $(L, \mathbf{D}_L)$ *where* L *is a distributive lattice with zero and* $\mathbf{D}_L$ *is a distinguished decent set of prime filters in* $\mathcal{PF}(L)$ *as objects,*
(2) *homomorphisms of lattices with zeros respecting the decent sets of prime filters and satisfying the condition of domination as morphisms.*

The category **ZLat** *may be identified with the full subcategory in* **ZLatD** *of objects satisfying* $\mathbf{D}_L = \mathcal{PF}(L)$*. Moreover, we have the category* **HAD** *of Heyting algebras with decent (i.e., constructibly dense) sets and homomorphisms of Heyting algebras respecting the decent sets.*

2.2. Categories of Spectral-like Spaces

Definition 8 ([6,7])**.** *An* object of **SpecBD** *is a system* $((X, \tau_X), CO_s(X), X_d)$ *where* (X, τ_X) *is a spectral space,* $CO_s(X)$ *is a bornology in the bounded lattice* $CO(X)$ *and* X_d *(a* decent lump*) satisfies the following conditions:*

(1) $X_d \subseteq \bigcup CO_s(X)$*,*
(2) $R_d : CO(X) \ni A \mapsto A \cap X_d \in CO(X) \cap_1 X_d$ *is an isomorphism of lattices,*
(3) $CO(X) \cap_1 X_d = (CO_s(X) \cap_1 X_d)^o \subseteq \mathcal{P}(X_d)$*.*

A morphism *from* $((X, \tau_X), CO_s(X), X_d)$ *to* $((Y, \tau_Y), CO_s(Y), Y_d)$ in **SpecBD** *is such a spectral mapping between spectral spaces* $g : (X, \tau_X) \to (Y, \tau_Y)$ *that:*

(a) *satisfies the* condition of boundedness $\forall A \in CO_s(X)\, \exists B \in CO_s(Y) \quad g(A) \subseteq B$*,*
(b) *respects the decent lump:* $g(X_d) \subseteq Y_d$*.*

We have the full subcategory **SpecB** *of objects satisfying* $X_d = \bigcup CO_s(X)$*. The category* **SpecD** *may be identified with the subcategory in* **SpecBD** *of objects satisfying* $CO_s(X) = CO(X)$*. We also have the category* **HSpecD** *of Heyting spectral spaces ([20]) with decent subsets and spectral mappings respecting the decent subsets.*

Definition 9 ([6])**.** *We have the category* **uSpec** *of up-spectral spaces and spectral mappings. The category* $\mathbf{uSpecD}^s$ *has*

(1) *pairs* $((X, \tau_X), X_d)$ *where* (X, τ_X) *is an up-spectral space and* X_d *is a distinguished decent subset of* X *as objects,*
(2) *bounded strongly continuous mappings respecting the decent subsets as morphisms.*

The category $\mathbf{uSpec}^s$ *may be identified with the full subcategory in* $\mathbf{uSpecD}^s$ *generated by objects satisfying* $X_d = X$*.*

2.3. Main Equivalences

Theorem 1 ([6,7])**.** *We have the following equivalences:*

1. *The categories* $\mathbf{LSS}_0$, $\mathbf{LatBD}^{op}$ *and* **SpecBD** *are equivalent.*
2. *The categories* $\mathbf{SS}_0$, $\mathbf{LatD}^{op}$ *and* **SpecD** *are equivalent.*
3. *The categories* **uSpec** *and* **SpecB** *are equivalent.*
4. *The categories* $\mathbf{uSpec}^s$ *and* **ZLat** *are dually equivalent.*
5. *The categories* $\mathbf{LSS}_0^s$, $\mathbf{ZLatD}^{op}$ *and* $\mathbf{uSpecD}^s$ *are equivalent.*
6. *The categories* **HSS**, **HSpecD** *and* $(\mathbf{HAD})^{op}$ *are equivalent.*

A version of Hofmann-Lawson duality for locally small spaces also exists ([48]).

2.4. The Spectralification Method and Consequences

In analysing small or locally small spaces, the spectral spaces ([20,31]) are especially helpful. This is achieved by the standard spectralifications, formally introduced in Section 5 of [7] (In particular, spectralifications of a Kolmogorov topological space may be constructed by choosing lattice bases of the topology). Theorem 1 can be seen as an extension of the method of taking the real spectrum or the o-minimal spectrum.

Corollaries on spaces: A Kolmogorov small space is essentially a constructibly dense subset of a spectral space, while a Kolmogorov locally small space is essentially a constructibly dense subset of an up-spectral space ([6]).

Corollaries on mappings: Bounded continuous mappings between T_0 locally small spaces are restrictions of spectral mappings between up-spectal (or just spectral) spaces to some constructibly dense subsets ([6]). Open continuous definable mappings between definable spaces over o-minimal structures are, in particular, Heyting continuous as mappings between Heyting small spaces ([7]).

3. Conclusions

Since families of sets closed under only finite unions are common in mathematics, a new branch of general topology (in the spirit of Engelking [49]), considering the above or new kinds of tame spaces and relevant mappings between them, is possible. We have initiated the development of such a branch by showing the above equivalences, while the usual topology gives only spectral reflections (see [20] (Chapter 11)). The use of smopologies should be helpful in such areas of mathematics as: the generalisations of o-minimality and other parts of model theory (especially where definable topologies are used), algebraic geometry, and analytic geometry.

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