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On Strong Approximation in Generalized Hölder and Zygmund Spaces †

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Abstract: The strong approximation of a function is a useful tool to analyze the convergence of its Fourier series. It is based on the summability techniques. However, unlike matrix summability methods, it uses non-linear methods to derive an auxiliary sequence using approximation errors generated by the series under analysis. In this paper, we give some direct results on the strong means of Fourier series of functions in generalized Holder and Zygmund spaces. To elaborate its use, we deduce some corollaries and a discussion follows the results.

Keywords: approximation; strong means; summability; hölder; zygmund

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1. Introduction

The strong approximation rather providing approximations is a tool of analysis. It is based on the summability techniques. However, unlike matrix summability methods, it uses non-linear methods to derive an auxiliary sequence using approximation errors generated by the series under analysis. This auxiliary sequence is further used to analyze the convergence properties of the series. To know more about the development of strong approximation methods, one can see the articles by Hyslop [1], and Mittal and Kumar [2]. They give simple settings of strong approximation along with some comparison results.

2. Preliminaries

The classical $L^p[0,2\pi]$ spaces define the foundations of Fourier analysis. An $L^p[0,2\pi]$, $1 \le p \le \infty$, space contains 2π-periodic, Lebesgue integrable functions, which have finite norm denoted by $\|\cdot\|_p$, and defined by

$$\|f\|_p = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^p dx\right)^{1/p}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{0\le x<2\pi}|f(x)|, & p = \infty. \end{cases}$$

To measure the smoothness of the functions, we use moduli of smoothness. For $f \in L^p[0,2\pi]$, the r^{th}-order modulus of smoothness $\omega_r(f;t)_p$ is defined by

$$\omega_r(f;t)_p = \sup_{0<h\le t}\|\Delta_h^r f(\cdot)\|_p$$

where $\Delta_h^r f(x) = \sum_{k=0}^{r}(-1)^k\binom{r}{k}f(x+(r-k)h)$ denotes r^{th} order forward difference of f at x with step-size h. The most basic spaces, which encode the smoothness properties of the functions, are the Hölder spaces. For the periodic functions, they were first introduced for the space of 2π-periodic continuous functions [3] (p. 51). Later, Das et al. [4] extended them to $L^p[0, 2\pi]$-spaces. They defined the Hölder spaces $H_\alpha(L^p)$, $0 < \alpha \leq 1$, to contain functions $f \in L^p[0, 2\pi]$ such that $\omega_1(f;t)_p = O(t^\alpha)$. The norm for $f \in H_\alpha(L^p)$ is given by $\|f\|_{H_\alpha(L^p)} = \|f\|_p + |f|_{H_\alpha(L^p)}$, where

$$|f|_{H_\alpha(L^p)} = \sup_{t>0} \frac{\omega_1(f;t)_p}{t^\alpha}.$$

As a further generalization, Das et al. [5] generalized Hölder spaces, $H_\alpha(L^p)$ to $H_\omega(L^p)$-spaces where $\omega : [0, \infty) \to [0, \infty)$ is a non-decreasing function with $\lim_{t\to 0^+} \omega(t) = 0$. The $H_\omega(L^p)$-spaces contain $f \in L^p[0, 2\pi]$ such that $\omega_1(f;t)_p = O(\omega(t))$. For $f \in H_\omega(L^p)$, the norm is defined by $\|f\|_{H_\omega(L^p)} := \|f\|_p + |f|_{H_\omega(L^p)}$, where

$$|f|_{H_\omega(L^p)} = \sup_{t>0} \frac{\omega_1(f;t)_p}{\omega(t)}.$$

The Zygmund spaces are defined using second order modulus of smoothness. The Zygmund spaces $Z_\alpha(L^p)$, $0 < \alpha \leq 2$, are defined to contain the functions $f \in L^p[0, 2\pi]$ such that $\omega_2(f;t)_p = O(t^\alpha)$. The norm for $f \in Z_\alpha(L^p)$ is defined by $\|f\|_{Z_\alpha(L^p)} = \|f\|_p + |f|_{Z_\alpha(L^p)}$, where

$$|f|_{Z_\alpha(L^p)} = \sup_{t>0} \frac{\omega_2(f;t)_p}{t^\alpha}.$$

The Zygmund spaces $Z_\alpha(L^p)$ can be generalized in the same way as Hölder spaces $H_\alpha(L^p)$. Let $\omega : [0, \infty) \to [0, \infty)$ be a non-decreasing function with $\lim_{t\to 0^+} \omega(t) = 0$. Then $Z_\omega(L^p)$ generalizes $Z_\alpha(L^p)$-spaces by the requirement $\omega_2(f;t)_p = O(\omega(t))$ for $f \in L^p[0, 2\pi]$. The norm for $f \in Z_\omega(L^p)$ is defined by $\|f\|_{Z_\omega(L^p)} = \|f\|_p + |f|_{Z_\omega(L^p)}$, where

$$|f|_{Z_\omega(L^p)} = \sup_{t>0} \frac{\omega_2(f;t)_p}{\omega(t)}.$$

For $\omega(t) = t^\alpha$, $0 < \alpha \leq 1$, the generalized Hölder space $H_\omega(L^p)$ becomes Hölder space $H_\alpha(L^p)$. Similarly, for $\omega(t) = t^\alpha, 0 < \alpha \leq 2$, the generalized Zygmund space $Z_\omega(L^p)$ coincides with Zygmund space $Z_\alpha(L^p)$. Because of the fact that $\omega_2(f;t)_p \leq 2\omega_1(f;t)_p$, the Hölder spaces are the subsets of corresponding Zygmund spaces. However, as the Zygmund norm is not same as Hölder norm on Hölder spaces, the Zygmund spaces do not generalize the Hölder spaces.

Let f be a 2π-periodic Lebesgue integrable function. Then, the partial sums $S_n(f;x), n = 0, 1, 2, \ldots$ of the trigonometric Fourier series of f can be written as

$$S_n(f;x) = \frac{1}{2\pi} \int_0^\pi (f(x+t) + f(x-t)) D_n(t) dt, \; n = 0, 1, 2, \ldots,$$

where $D_n(t)$ denotes the Dirichlet kernel of order n given by

$$D_n(t) = 1 + 2\sum_{k=1}^{n} \cos kt = \begin{cases} 2n+1, & t = 2m\pi, \; m \in \mathbb{Z}, \\ \dfrac{\sin\left(\dfrac{(2n+1)t}{2}\right)}{\sin\left(\dfrac{t}{2}\right)}, & \text{otherwise.} \end{cases}$$

The properties of the Dirichlet kernels can be found in [6] (p. 235). $R_n(f;x)$, the n^{th} residual in the approximation of f by partial sums $S_n(f;x)$, can be written as following:

$$R_n(f;x) := S_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \Delta_t^2 f(x-t) D_n(t)dt, \ n = 0,1,2,\ldots.$$

The summability techniques are an extension of the notion of convergence of a series in which we try to attach a limit to a non-convergent sequence. The summability methods may also increase the rate of convergence of an already convergent series. A summability matrix $T = (a_{n,k}), \ n,k = 0,1,2,\ldots$ is called regular summability matrix if it satisfies:

(i) $\lim_{n\to\infty} a_{n,k} = 0$,
(ii) $\sum_{k=0}^{\infty}|a_{n,k}| \leq M \geq 0, \forall n$,
(iii) $\sum_{k=0}^{\infty} a_{n,k} = 1, \forall n$.

Let $T = (a_{n,k}), n,k = 0,1,2,\ldots$ be an infinite-dimensional, lower triangular summability matrix. Then the sequence

$$T_n(f;x) = \sum_{k=0}^{n} a_{n,k} S_k(f;x), \ n = 0,1,2,\ldots,$$

defines the T-means of the trigonometric Fourier series of f. The difference $\rho_n(f;x) := T_n(f;x) - f(x)$ denotes the n^{th} residual in the approximation of f by T-means of the Fourier series. If

$$\sum_{k=0}^{n} a_{n,k} = 1, \ n = 0,1,2,\ldots, \tag{1}$$

then we can write

$$\rho_n(f;x) = \sum_{k=0}^{n} a_{n,k} R_k(f;x) = \frac{1}{2\pi}\int_0^{\pi} \Delta_t^2 f(x-t) K_n(t)dt,$$

where $K_n(t)$ is the kernel generated by T and defined by $K_n(t) := \sum_{k=0}^{n} a_{n,k} D_k(t), \ n = 0,1,2,\ldots$. If $T = (a_{n,k})$ is such that $a_{n,k} \geq 0, \ n,k = 0,1,2,\ldots$, then for $\lambda > 0$, the T-strong means of the Fourier series are defined by

$$U_n(f,\lambda,x) = \left(\sum_{k=0}^{n} a_{n,k} |R_k(f;x)|^{\lambda}\right)^{1/\lambda}.$$

In this paper, we present estimates of $U_n(f,\lambda,x)$ in $H_\omega(L^p)$ and $Z_\omega(L^p)$ spaces. These estimates help in gaining insights of the approximation error in Fourier approximation.

3. Results

First, we prove some auxiliary results as lemmas to make the proof of main results concise.

Lemma 1. *Let $D_k(t)$ be the k^{th} Dirichlet kernel. Then*

(i) $|D_k(t)| = O(k+1), \ t \in [0,\pi]$.
(ii) $|D_k(t)| = O(1/t), \ t \in (0,\pi]$.

The lemma can be proved easily.

Lemma 2. *Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of 2π-periodic Lebesgue measurable functions. Then, for any $p \geq 1$ and $0 < \lambda \leq p$*

$$\left\| \left(\sum_{k=0}^{\infty} |f_k(x)|^{\lambda} \right)^{1/\lambda} \right\|_p \leq \left(\sum_{k=0}^{\infty} \|f_k\|_p^{\lambda} \right)^{1/\lambda}.$$

Using generalized Minkowski inequality, the lemma can proved easily.

Now, we present the main results.

Theorem 1. *Let $T = (a_{n,k})$ be a summability matrix such that $a_{n,k} \geq 0,\ n,k = 0,1,2,\ldots$. Then, for $f \in H_{\omega}(L^p)$ and $0 < \lambda \leq p$*

$$\|U_n(f,\lambda,x)\|_{H_{\omega}(L^p)} = O\left(\sum_{k=0}^{n} a_{n,k} \left(\omega\left(\frac{\pi}{k+1}\right) + \int_{\frac{\pi}{k+1}}^{\pi} \frac{\omega(t)}{t} dt \right)^{\lambda} \right)^{1/\lambda},$$

provided f satisfies the following conditions:

(i) $\Delta_h^1 U_n(f,\lambda,x) = O\big(U_n(\Delta_h^1 f,\lambda,x)\big)$.

(ii) $\|\Delta_h^1 \Delta_t^2 f(x)\|_p = O\big(\omega_1(f;t)\|\Delta_h^1 f(x)\|_p\big)$.

Proof. By the definition of $\|\cdot\|_{H_{\omega}(L^p)}$, we have

$$\|U_n(f,\lambda,x)\|_{H_{\omega}(L^p)} = \|U_n(f,\lambda,x)\|_p + |U_n(f,\lambda,x)|_{H_{\omega}(L^p)}.$$

We first estimate $\|U_n(f,\lambda,x)\|_p$. Using Lemma 2,

$$\|U_n(f,\lambda,x)\|_p = \left\| \left(\sum_{k=0}^{n} a_{n,k} |R_k(f,x)|^{\lambda} \right)^{1/\lambda} \right\|_p \leq \left(\sum_{k=0}^{n} a_{n,k} \|R_k(f,x)\|_p^{\lambda} \right)^{1/\lambda}. \tag{2}$$

Using the generalized Minkowski inequality, Lemma 1 and definition of $R_k(f,x)$, we have

$$\begin{aligned}
\|R_k(f,x)\|_p &= \left\| \frac{1}{2\pi} \int_0^{\pi} \Delta_t^2 f(x-t) D_k(t) dt \right\|_p \\
&= O\left(\int_0^{\pi} \omega_2(f;t)_p |D_k(t)| dt \right) \\
&= O\left(\int_0^{\pi} \omega_1(f;t)_p |D_k(t)| dt \right) \\
&= O\left(\int_0^{\frac{\pi}{k+1}} \omega_1(f;t)_p |D_k(t)| dt + \int_{\frac{\pi}{k+1}}^{\pi} \omega_1(f;t)_p |D_k(t)| dt \right) \\
&= O\left(\omega\left(\frac{\pi}{k+1}\right) + \int_{\frac{\pi}{k+1}}^{\pi} \frac{\omega(t)}{t} dt \right).
\end{aligned} \tag{3}$$

From (2) and (3), we have

$$\|U_n(f,\lambda,x)\|_p = O\left(\sum_{k=0}^{n} a_{n,k} \left(\omega\left(\frac{\pi}{k+1}\right) + \int_{\frac{\pi}{k+1}}^{\pi} \frac{\omega(t)}{t} dt \right)^{\lambda} \right)^{1/\lambda}. \tag{4}$$

Now, we estimate $|U_n(f,\lambda,x)|_{H_\omega(L^p)}$. Using condition (i), we have

$$\|\Delta_h^1 U_n(f,\lambda,x)\|_p = O\Big(\|U_n(\Delta_h^1 f,\lambda,x)\|_p\Big) = O\left(\left\|\left(\sum_{k=0}^n a_{n,k}|R_k(\Delta_h^1 f,x)|^\lambda\right)^{1/\lambda}\right\|_p\right).$$

Using Lemma 2, definition of $R_k(f,x)$ and condition (ii), we have

$$\|\Delta_h^1 U_n(f,\lambda,x)\|_p = O\left(\sum_{k=0}^n a_{n,k}\left\|R_k(\Delta_h^1 f,x)\right\|_p^\lambda\right)^{1/\lambda} = O\left(\|\Delta_h^1 f(x)\|_p\left(\sum_{k=0}^n a_{n,k}\left(\int_0^\pi \omega_1(f;t)|D_k(t)|dt\right)^\lambda\right)^{1/\lambda}\right).$$

Following calculation in (3), we have

$$\|\Delta_h^1 U_n(f,\lambda,x)\|_p = O\left(\|\Delta_h^1 f(x)\|_p\left(\sum_{k=0}^n a_{n,k}\left(\omega\left(\frac{\pi}{k+1}\right)+\int_{\frac{\pi}{k+1}}^\pi \frac{\omega(t)}{t}dt\right)^\lambda\right)^{1/\lambda}\right).$$

Taking supremum for $0<h\le u$ on both sides

$$\omega_1(U_n(f,\lambda,x);u) = O\left(\omega_1(f;u)\left(\sum_{k=0}^n a_{n,k}\left(\omega\left(\frac{\pi}{k+1}\right)+\int_{\frac{\pi}{k+1}}^\pi \frac{\omega(t)}{t}dt\right)^\lambda\right)^{1/\lambda}\right).$$

Therefore,

$$\begin{aligned}|U_n(f,\lambda,x)|_{H^\omega(L^p)} &= \sup_{u>0}\frac{\omega_1(U_n(f,\lambda,x);u)_p}{\omega(u)}\\ &= O\left(\sup_{u>0}\frac{\omega_2(f;u)_p}{\omega(u)}\left(\sum_{k=0}^n a_{n,k}\left(\omega\left(\frac{\pi}{k+1}\right)+\int_{\frac{\pi}{k+1}}^\pi \frac{\omega(t)}{t}dt\right)^\lambda\right)^{1/\lambda}\right)\\ &= O\left(\sum_{k=0}^n a_{n,k}\left(\omega\left(\frac{\pi}{k+1}\right)+\int_{\frac{\pi}{k+1}}^\pi \frac{\omega(t)}{t}dt\right)^\lambda\right)^{1/\lambda}.\end{aligned} \tag{5}$$

Combining (4) and (5), we have

$$\|U_n(f,\lambda,x)\|_{H_\omega(L^p)} = O\left(\left(\sum_{k=0}^n a_{n,k}\left(\omega\left(\frac{\pi}{k+1}\right)+\int_{\frac{\pi}{k+1}}^\pi \frac{\omega(t)}{t}dt\right)^\lambda\right)^{1/\lambda}\right),$$

which completes the proof of the theorem. □

Depending on the value of λ, condition (i) in Theorem 1 can be relaxed. More precisely, the following holds.

Corollary 1. *Let $T = (a_{n,k})$ be a summability matrix such that $a_{n,k} \geq 0,\ n,k = 0,1,2,\ldots$. Then, for $f \in H_\omega(L^p)$ and $1 \leq \lambda \leq p$*

$$\|U_n(f,\lambda,x)\|_{H_\omega(L^p)} = O\left(\sum_{k=0}^{n} a_{n,k}\left(\omega\left(\frac{\pi}{k+1}\right) + \int_{\frac{\pi}{k+1}}^{\pi} \frac{\omega(t)}{t}dt\right)^{\lambda}\right)^{1/\lambda}$$

provided f satisfies $\|\Delta_h^1\Delta_t^2 f(x)\|_p = O\big(\omega_1(f;t)\|\Delta_h^1 f(x)\|_p\big)$.

Proof. In the light of the Minkowski inequality for sequence spaces, the condition (i) in Theorem 1 holds for $\lambda \geq 1$. Then, the corollary follows from Theorem 1. □

Since, for $\omega(t) = t^\alpha, 0 < \alpha \leq 1$, $H_\omega(L^p)$-space reduces to $H_\alpha(L^p)$-space, we have the following corollary for $f \in H_\alpha(L^p)$.

Corollary 2. *Let $T = (a_{n,k})$ be a summability matrix such that $a_{n,k} \geq 0,\ n,k = 0,1,2,\ldots$ and satisfies (1). Then, for $f \in H_\alpha(L^p)$ and $0 < \lambda \leq p$*

$$\|U_n(f,\lambda,x)\|_{H_\alpha(L^p)} = O(1),$$

provided f satisfies the following conditions:

(i) $\Delta_h^1 U_n(f,\lambda,x) = O\big(U_n(\Delta_h^1 f,\lambda,x)\big)$.

(ii) $\|\Delta_h^1\Delta_t^2 f(x)\|_p = O\big(\omega_1(f;t)\|\Delta_h^1 f(x)\|_p\big)$.

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