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Control of Incomplete Fractional Punishment In Optional Public Goods Game

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INTRODUCTION & AIM

The use of the Public Goods Game as a proxy for collaborative community efforts is widespread among many academic circles. The main question revolves around how to solve the free-rider problem in the most cost-effective way possible, and this tends to involve the optimal use of incentives.
This work presents how a reinterpretation of the controlled punishment used in Botta 2021 and Grau 2022 based of the findings of Botta 2024 allows for a more realistic punishment strategy, that at the same time may be more cost-effective.
The system analyzed is as described in Botta 2021, using the replicator dynamics for the Optional Public Goods Game with only one difference: fractional punishment variable “d” is now replaced by “v” which is the product of the fraction of punished free-riders and how much of their expected benefit is taken from them as punishment.

METHODS

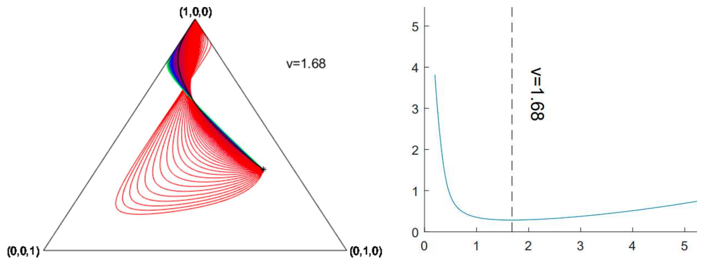
Let $e(t) \coloneqq [x(t)-1, y(t), z(t)]$ be the state error when x, y, z are the frequencies of cooperators, free-riders and independents respectively. Then cost functions similar to equation (1) were used to analyze and compare the reduction in costs obtained by allowing the control variable to go beyond 1 with the previously mentioned mechanism.

$$J = \frac{\alpha_1}{2} e_f^T e_f + \int_0^{t_f} \left[\frac{\alpha_2}{2} e^T e + \frac{\alpha_3}{2} v^2 + \frac{\alpha_3}{2} v^2 + \frac{\alpha_4}{2} v^2 y^2\right] d\tau \ \ldots (1)$$

The Python package GEKKO was used to solve the optimization problem that minimizes equation (1) for each case.

RESULTS & DISCUSSION

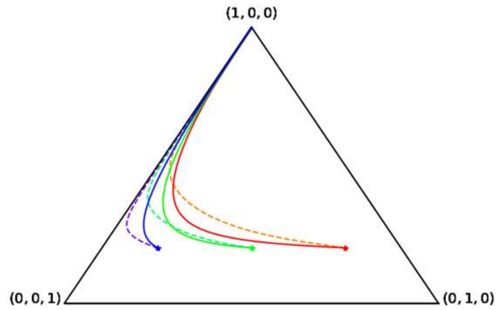
Preliminary results like the one in Figure 1 show that going beyond 1 for constant punishment strategies already allows for better results in some cases.



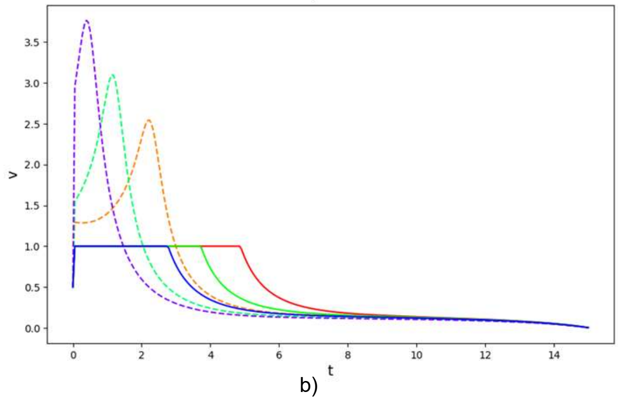
a) b)

Figure 1: Given weights [0, 0.6, 0.001, 0.399] for cost function (1), a) shows the trajectories in state space for constant control values of v with starting state [0.35, 0.55, 0.1] going from red to blue and then green as v increases. The black trajectory indicates the one with the lowest cost and said value is annotated. And b) shows the values of cost function J as a function of the constant control values that v takes. Again, the minimum is shown in black with a dashed vertical line as a marker of the spot.

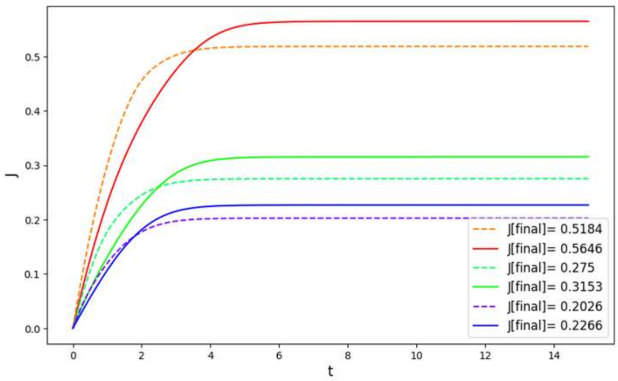
However, although choosing a constant control strategy that minimizes costs is interesting in itself because of the simplicity of the approach, an optimal strategy that changes through time has been shown to be more cost-efficient [2].
To further explore this, a representative cost function is taken to show the difference between optimal control strategies with an upper bound at 1 and those without an upper bound. This is shown in Figure 2.
Although there are clear gains made from this, it needs to be pointed out that even with perfect locating of all free riders (fraction of punished free-riders equal to 1), the fines they may acquire as punishment may be too high for them to pay, indirectly causing the complete exclusion of these individuals from the system.



a)



b)



c)

Figure 2: Given weights [0, 0.2, 0.001, 0.7 99] for cost function (1), a) shows the trajectories in state space for initial states [0.2, 0.65, 0.15], [0.2, 0.4, 0.4] and [0.2, 0.15, 0.65] using bounded (solid lines) and unbounded (dashed lines) optimal controllers to minimize the cost function. b) and c) shows the corresponding curves for control variable v and costs J in the analyzed time period. The last image shows the final values of each cost curve, allowing us to compare the paired trajectories.

CONCLUSION AND FUTURE WORK

The incomplete punishment strategy allows to make further gains from what was previously achieved using only fractional punishment. This however needs to be scrutinized just as any other punishment strategy in order to bring it closer to a real scenario, given that at the time there is no cutoff to how big the fines an individual could acquire are. Future works will focus on that.

REFERENCES

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[2] Grau, J., Botta, R., & Schaerer, C. (2022). Optimal Control of Fractional Punishment in Optional Public Goods Game. In Proceeding Series of the Brazilian Society of Computational and Applied Mathematics (pp. 1-2). SBMAC - Sociedade de Matemática Aplicada e Computacional.
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