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Application of relative entropy in finding the minimal equivalent martingale measure (A note on MEMM)

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Overview

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Introduction

Minimal entropy martingale measure (MEMM) and geometric Levy process has been introduced as a pricing model for the incomplete financial market. This model has many good properties and is applicable to very wide classes of underlying asset price processes. MEMM is the nearest equivalent martingale measure to the original probability in the sense of Kullback-Leibler distance and is closely related to the large deviation theory .Those good properties has been explained. MEMM is also justified for option pricing problem when the risky underlying assets are driven by Markov-modulated Geometric Brownian Motion and Markov-modulated exponential Levy model.

Introduction

The equivalent martingale measure method is one of the most powerful methods in the option pricing theory. If the market is complete, then the equivalent martingale measure is unique. On the other hand, in the incomplete market model there are many equivalent martingale measures. So we have to select one equivalent martingale measure (EMM) as the suitable martingale measure in order to apply the martingale measure method

Introduction

In this present, MEMM method is reviewed. Our paper organizes as follows: Section two presents the main idea of our paper. Geometric Lévy process and minimal entropy martingale measure pricing models is stated in section two. Section three considers Option pricing and MEMM under regime switching. The forth section is related to some other application of MEMM. Finally have been stated conclusion of the paper and proposes some topics for further investigation.

Geometric Lévy process and MEMM pricing models

A pricing model consists of the following two parts:
(A) The price process *St* of the underlying asset.
(B) The rule to compute the prices of options.
For the part (A) we adopt the geometric Lévy processes, so the part (A) is reduced to the selecting problem of a suitable class of the geometric Lévy processes. For the second part (B) we adopt the martingale measure method. then the price of an option X is given by:

$$e^{-rt}E_Q[X]$$

Geometric Lévy processes(GLP)

The price process S_t of a stock is assumed to be defined as what follows. We suppose that a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t; 0 \geq t \geq T\}$ are given, and that the price process $S_t = S_0 e^{Z_T}$ of a stock is defined on this probability space and given in the form

$$S_t = S_0 e^{Z_T} \qquad 0 \leq t \leq T$$

Where Z_t is a lévy process. Such a process S_t is named the geometric lévy process (GLP) and denoted the generating triplet of Z_t by . $(\sigma^2, v(dx), b)$

Simple return process and compound return process

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the GLP has two kinds of representation such that

$$S_t = S_0 e^{Z_T} = S_0 \mathcal{E}(\tilde{Z})_t$$

Where $\mathcal{E}(\tilde{Z})_t$ is the Doléans-Dade exponential (or stochastic exponential) of Z_t.

The processes Z_t and $\tilde{Z}_t$ are candidates for the risk process. It is shown that

$$\Delta\tilde{Z}_k^{(n)} = \frac{\Delta S_k^{(n)}}{S_{k-1}^{(n)}} \quad , \quad \Delta Z_k^{(n)} = \log\left(1 + \frac{\Delta S_k^{(n)}}{S_{k-1}^{(n)}}\right)$$

Thus, $\Delta\tilde{Z}_k^{(n)}$ is the simple return process of $S_k^{(n)}$ and $\Delta Z_k^{(n)}$ is the increment of log-returns and it is called the compound return process of $S_k^{(n)}$.

Esscher transforms

Let $R_t, 0 \leq t \leq T$ be a stochastic process. Then the Esscher transformed measure of P by the risk process R_t and the index process h_s is the probability measure of $P^{(ESS)}_{R_{[0,T]},h}$ defined by

$$\frac{P^{(ESS)}_{R_{[0,T]},h_{[0,T]}}}{dP}\Big|_{\mathcal{F}} = \frac{e^{\int_0^T h_s dR_s}}{E[e^{\int_0^T h_s dR_s}]}$$

This measure transformation is called the Esscher transform by the risk process R_t and the index process h_s
.

Esscher Martingale Measure (ESMM)

In the above definitions, if the index process is chosen so that the $P^{(ESS)}_{R_{[0,T]},h_{[0,T]}}$ is a martingale measure of S_t ,
then , $P^{(ESS)}_{R_{[0,T]},h_{[0,T]}}$ is called the Esscher transformed martingale measure of

Minimal entropy martingale measure (MEMM)

If an equivalent martingale measure P^* satisfies

$$H(P^*|P) \le H(Q|P) \qquad \forall Q\text{: equivalent martingale measure;}$$

then is called the minimal entropy martingale measure (MEMM) of S_T Where $H(Q|P)$ is the relative entropy of Q with respect to P

$$H(Q|P) = \begin{cases} \int_\Omega \log\left[\frac{dQ}{dP}\right] dQ \text{ , } if\ Q \ll P \\ \infty, \qquad otherwise, \end{cases}$$

Minimal entropy martingale measure (MEMM)

Proposition: The simple return Esscher transformed martingale measure $P^{(ESS)}_{Z_{[0,T]}}$ of S_t is the minimal entropy martingale measure (MEMM) of S_t .

Remark: The uniqueness and existence theorems of ESMM and MEMM for geometric Lévy processes is proved in [3].

Comparison of ESMM and MEMM

a)Corresponding risk process: The risk process corresponding to the ESMM is the compound return process, and the risk process corresponding to the MEMM is the simple return process. The simple return process seems to be more essential in the relation to the original process rather than the compound return process.

b) Existence condition: For the existence of ESMM, and MEMM, the following condition respectively is necessary.

$$\int_{\{|x|>1\}} |(e^x - 1)e^{h^* x}|\nu(dx) < \infty$$

$$\int_{\{|x|>1\}} |(e^x - 1)e^{\theta^*(e^x - 1)}|\nu(dx) < \infty$$

This means that the MEMM may be applied to the wider class of models than the ESMM. This difference does work in the stable process cases

Comparison of ESMM and MEMM

c) Corresponding utility function: The ESMM is corresponding to power utility function or logarithm utility function. On the other hand the MEMM is corresponding to the exponential utility function. the MEMM is very useful when one studies the valuation of contingent claims by (exponential) utility indifference valuation.

Properties special to MEMM

a) Minimal distance to the original probability:
The relative entropy is very popular in the field of information theory, and it is called Kullback-Leibler Information Number or Kullback-Leibler distance. Therefore we can state that the MEMM is the nearest equivalent martingale measure to the original probability P in the sense of Kullback-Leibler distance.

b) Large deviation property:
The large deviation theory is closely related to the minimum relative entropy analysis, and the Sanov's theorem or Sanov property is well-known. This theorem says that the MEMM is the most possible empirical probability measure of paths of price process in the class of the equivalent martingale measures.

Properties special to MEMM

c)convergence question:

Several authors have proved in several settings and with various techniques that the minimal entropy martingale measure is the limit, as $p \searrow 1$, of the so-called p-optimal martingale measures obtained by minimizing the f-divergence associated to the function $f(y) = y^p$.

Option pricing and Esscher transform under regime switching

In [4] is considered the option pricing problem when the risky underlying assets are driven by Markov-modulated Geometric Brownian Motion (GBM). That is, the market parameters, for instance, the market interest rate, the appreciation rate and the volatility of the underlying risky asset, depend on unobservable states of the economy which are modeled by a continuous-time Hidden Markov process. They adopt a regime switching random Esscher transform to determine an equivalent martingale pricing measure. As in Miyahara [2], they can justify their pricing result by the minimal entropy martingale measure (MEMM).

The model

We assume that the states of the economy are modeled by a continuous-time hidden Markov Chain process $\{X_t\}_{t\in T}$ on (Ω, F, P) with a finite state space $X := (x_1, x_2, \ldots, x_N)$

we have the following semi-martingale representation theorem for $\{X_t\}_{t\in T}$

$$X_t = X_0 + \int_0^t A(s)X_s ds + M_t$$

Where $\{M_t\}_{t\in T}$ is an $\mathcal{R}^N$-valued martingale increment process with respect to the filtration generated by $\{X_t\}_{t\in T}$

The model

The dynamics of the stock price process $\{S_t\}_{t\in T}$ are then given by the following Markov-modulated Geometric Brownian Motion:

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, S_0 = s$$

Let Z_t denote the logarithmic return $ln(S_t/S_0)$ from over the interval $[0,t]$.

Where

$$Z_t = \int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2\right) ds + \int_0^t \sigma_s dW_s$$

The model

the regime switching Esscher transform $Q_\theta \sim P$ on $\mathcal{G}_t$ with respect to a family of parameters $\{\theta_s\}_{s\in[0,t]}$ is given by:

$$\frac{dQ_\theta}{dP}\Big|_{\mathcal{G}_t} = \frac{\exp\left(\int_0^t \theta_s dZ_s\right)}{E_P\left[\exp\left(\int_0^t \theta_s dZ_s\right)\middle|\mathcal{F}_t^X\right]}, t \in T$$

The Radon-Nikodim derivative of the regime switching Esscher transform is given by:

$$\frac{dQ_\theta}{dP}\Big|_{\mathcal{G}_t} = \exp\left(\int_0^t \theta_s \sigma_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 \sigma_s^2 ds\right)$$

The model

Proposition: Suppose there exists a β_t such that the following equation is satisfied:

$$\beta_t = \frac{r_t - \mu_t}{\sigma_t^2}$$

Let Q^* be a probability measure equivalent to the measure P on $\mathcal{G}_t$ defined by the following Radon-Nikodym derivative:

$$\frac{dQ^*}{dP}\Big|_{\mathcal{G}_t} := \exp\left(\int_0^t \beta_s \sigma_s dW_s - \frac{1}{2}\int_0^t \beta_s^2 \sigma_s^2 ds\right)$$

Then,

1. Q^* is well defined and uniquely determined by the above Radon-Nikodym derivative,
2. Q^* is the MEMM for the Markov-modulated GBM.

Conclusions

As we have seen, the MEMM has many good properties and seems to be superior to ESSMM from the theoretical point of view. And we can say that the [GLP & MEMM] model, which has been introduced in [8], is a strong candidate for the incomplete market model.
Developing method is found to price options when the risky underlying assets are driven by Markov-modulated Geometric Brownian Motion (GBM) based on a modification of the random Esscher transform by Siu et al. [9], namely the regime switching random Esscher transform. The choice of this martingale pricing measure is justified by the minimization of the relative entropy. Finally may explore the applications of these models to other types of exotic options or hybrid financial products, such as barrier options Asian options, game options and option-embedded insurance products, etc.

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Thank you for your attention