Black hole/Bose gas duality and statistical description of entropy

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Introduction

The main objective of our study is finding black hole solutions that do not violate the third law of thermodynamics. First, we examine possibility of finding such solution in Einstein-Maxwell-Dilaton model and then study it's thermodynamics properties. Afterward, we find the duality between Bose gas and following black hole solution. This establishes statistical description of black hole entropy, for various dimension and α anisotropy parameters, in contrast to Strominger-Vaffa approach [1], which provided microscopic origin for entropy only for extremal black hole.

Third Law of Thermodynamics

1. Planck formulation. When temperature falls to absolute zero, the entropy tends to a universal constant (which can be taken to be zero) M. Planck, Thermodynamik. 1911.

$$S \to 0 \quad \text{as} \quad T \to 0$$

2. Einstein formulation [2]. As the temperature falls to absolute zero, the entropy of any substance remains finite. this implies that S is finite at $T \to 0$.
3. Wald formulation[3] "The third law of ordinary thermodynamics asserts that the entropy, S, of a system must go to zero (or a "universal constant") as its temperature, T, goes to zero."

The third law of thermodynamics implies that as the temperature of an isolated system decreases, its energy and entropy—which measure disorder or motion—should also decrease. If entropy remained, it would imply motion without motion, creating as we call this problem an entropic paradox or a contradiction of the third law.

Problem with the Schwarzschild black hole

The metric of the Schwarzschild black hole in $D = 4$ spacetime is

$$ds^2 = -\left(1 - \frac{r_h}{r}\right)dt^2 + \frac{dr^2}{(1 - \frac{r_h}{r})} + r^2 d\Omega_2^2, \qquad (1)$$

The Hawking temperature and the Bekenstein-Hawking entropy are

$$T = \frac{1}{8\pi G_4 M}, \qquad S = \frac{r_h^2}{4G_4} = \frac{1}{16\pi G_4 T^2}. \qquad (2)$$

1. S goes to infinity as $T \to 0$ and we see a violation of the third law in Planck's formulation. [3]
2. black hole explosion problem. For $M \to 0$ Hawking temperature goes as $T \to \infty$.[4]

Electric Lifshitz case

The holographic model is supported by Einstein-dilaton-Maxwell action in the form [5,6,7]

$$S = \frac{1}{16\pi G}\int d^5x\sqrt{-g}[R - \Lambda - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}e^{\lambda\phi}F_{\mu\nu}F^{\mu\nu}] \qquad (3)$$

where $F_{\mu\nu}$ Maxwell field tensor field defined via vector potenial A_μ as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad (4)$$

Consider ansatz with only electric component of vector potential in form

$$A_\mu = (A_t(z), 0, 0, 0, 0) \qquad (5)$$

The metric ansatz with α anisotropy parameter

$$ds^2 = L^2[-z^{2\alpha}f(z)dt^2 + z^2dx^2 + z^2dy_1^2 + z^2dy_2^2 + \frac{dz^2}{z^2 f(z)}] \qquad (6)$$

Equation of Motion

The Maxwell and Scalar equations

$$\frac{\alpha A_t'(z)}{z} - \lambda A_t'(z)\phi'(z) - \frac{4A_t'(z)}{z} - A_t''(z) = 0 \qquad (7)$$

$$\frac{\lambda z^{-2\alpha}A_t'(z)^2 e^{\lambda\phi(z)}}{2L^2 f(z)} + \frac{\alpha\phi'(z)}{z} + \frac{f'(z)\phi'(z)}{f(z)} + \frac{4\phi'(z)}{z} + \phi''(z) = 0 \qquad (8)$$

And 3 independent gravitational equations

$$\frac{z^{-2\alpha}A_t'(z)^2 e^{\lambda\phi(z)}}{L^2 f(z)} + \frac{6f'(z)}{zf(z)} + \frac{2L^2\Lambda}{z^2 f(z)} + \frac{24}{z^2} + \phi'(z)^2 = 0 \qquad (9)$$

$$\frac{2\alpha^2 f(z)}{z^2} - \frac{z^{-2\alpha}A_t'(z)^2 e^{\lambda\phi(z)}}{2L^2} + \frac{2\left(\alpha^2 + 2\alpha + 3\right)f(z)}{z^2} + \frac{(3\alpha+5)f'(z)}{z} + \frac{1}{2}f(z)\phi'(z)^2 + cL^2\Lambda z^2 + f''(z) = 0 \qquad (10)$$

$$\frac{z^{-2\alpha}A_t'(z)^2 e^{\lambda\phi(z)}}{L^2 f(z)} + \frac{12\alpha}{z^2} + \frac{6f'(z)}{zf(z)} + \frac{2L^2\Lambda}{z^2 f(z)} + \frac{12}{z^2} - \phi'(z)^2 = 0 \qquad (11)$$

By solving this EOM system one can find that

$$f(z) = 1 - \left(\frac{z_h}{z}\right)^{3+\alpha}$$

With

$$\Lambda = -\frac{(\alpha+2)(\alpha+3)}{L^2}, \qquad \lambda = -\frac{\sqrt{6}}{\sqrt{\alpha-1}}, \qquad \phi(z) = \pm\sqrt{6(\alpha-1)}\log(z)$$

$$A_t(z) = \frac{qz^{\alpha+3}}{\alpha+3} \qquad q = \sqrt{2}\sqrt{(\alpha-1)(\alpha+3)}L$$

Thermodyanmics

The Hawking temperature

$$T_H = \frac{(\alpha+3)L^2 z_h^\alpha}{4\pi} \qquad (12)$$

$$S = 4\pi L^3 z_h^3 \qquad (13)$$

By expressing entropy in terms of T

$$S = \frac{(4\pi)^{\frac{3}{\alpha}+1}L^3 T^{3/\alpha}}{(\alpha+3)^{3/\alpha}} \qquad (14)$$

- NO BH explosion
- NO violation of 3LOT

Bose gas

Consider a Bose gas Free energy in a d-dimensional space

$$F_{BG} = \frac{1}{\beta}\sum_{\substack{k_i = 2\pi n_i/\mathcal{L} \\ i = 1,\dots,d \\ n_i = 1,2,\dots}} \ln\left(1 - e^{\beta(\mu - \sigma\,\varepsilon_\gamma(\vec{k}))}\right), \qquad (15)$$

where $\varepsilon_\gamma(\vec{k})$ is the energy of quasi-particles:

$$\varepsilon_\gamma(\vec{k}) = k^\gamma \equiv \left(\vec{k}^2\right)^{\gamma/2}, \qquad \vec{k} = \{k_1, \dots, k_d\}, \qquad (16)$$

and σ is a positive dimensional constant. Using spherical coordinates and integrating over spherical angles, we obtain for $\mu = 0$:

$$F_{BG} = \frac{\Omega_{d-1}}{\beta}\left(\frac{\mathcal{L}}{2\pi}\right)^d \int_0^\infty \ln\left(1 - e^{-\beta\sigma k^\gamma}\right)k^{d-1}\,dk, \qquad (17)$$

where $\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$, and k^γ represents the exponent of the radial momenta as defined in (16).

Entropy of Bose gas

The entropy is given by

$$S_{BG} = \beta^2\frac{\partial F_{BG}}{\partial\beta} = \left(1 + \frac{d}{\gamma}\right)\left(\frac{\mathcal{L}}{2\pi}\right)^d \frac{2\pi^{d/2}}{d\,\Gamma(d/2)}\left(\frac{T}{\sigma}\right)^{\frac{d}{\gamma}}\Gamma\left(\frac{d}{\gamma}+1\right)\zeta\left(\frac{d}{\gamma}+1\right).$$

Duality

Equalizing the entropy of the perfect Bose gas of quasi-particles and entropy of Lifshitz black brane we get an equality

$$(1 + \frac{d}{\gamma})\left(\frac{\mathcal{L}}{2\pi}\right)^d \frac{2\pi^{d/2}\Gamma\left(\frac{d}{\gamma}+1\right)}{d\Gamma(d/2)}\left(\frac{T}{\sigma}\right)^{\frac{d}{\gamma}}\zeta\left(\frac{d}{\gamma}+1\right) = \frac{(4\pi)^{\frac{3}{\alpha}+1}L^3T^{3/\alpha}}{(\alpha+3)^{3/\alpha}} \qquad (18)$$

We see that

$$T^{\frac{d}{\gamma}} = T^{3/\alpha} \quad \to \quad d = \frac{3\gamma}{\alpha}, \quad \alpha = \frac{3\gamma}{d} \qquad (19)$$

Example

1. $d = 3 \to \gamma = 2, \alpha = 2$
2. $d = 4 \to \gamma = 2, \alpha = 3/2$
3. $d = 4 \to \gamma = 4, \alpha = 3$

Conclusions & outlook

1. We resolved the 3LOT violation problem for anisotropic Lifshitz-like black holes via regulation by parameter α.
2. The duality between Lifshitz black hole and Bose gas was established for various dimensions and γ parameter of energy of quasi-particle.
3. The generalization of duality for arbitrary D-dim case and magnetic ansatz was done in [8]. In that case the duality condition holds

$$\frac{d}{\gamma} = \frac{D-2}{\alpha} \qquad (20)$$

Open questions:

1. Duality between rotating Bose gas in a trap
2. Duality between Bose gas with non-zero chemical potential

References

[1] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawking entropy," [arXiv:hep-th/9601029 [hep-th]].
[2] A. Einstein, Annalen Phys. **23** (1907) no.7, 371-384 doi:10.1002/andp.19073280713
[3] R. M. Wald, "The 'Nernst theorem' and black hole thermodynamics," Phys. Rev. D **56** (1997), 6467-6474
[4] S. W. Hawking, "Black hole explosions," Nature **248** (1974), 30-31
[5]M. Taylor, "Non-relativistic holography," [arXiv:0812.0530 [hep-th]].
[6]D. W. Pang, "A Note on Black Holes in Asymptotically Lifshitz Spacetime,", [arXiv:0905.2678 [hep-th]].
[7]I. Y. Aref'eva, K. Rannu and P. Slepov, "Holographic anisotropic model for light quarks with confinement-deconfinement phase transition," JHEP **06** (2021), 090,[arXiv:2009.05562 [hep-th]].
[8]I. Aref'eva, I. Volovich and D. Stepanenko, "Black brane/Bose gas duality and the third law of thermodynamics," Teor. Mat. Fiz. **222** (2025) no.2, 276-284 [arXiv:2411.01778 [hep-th]].

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