Dynamical system study of Interacting Cosmological Dark Sector in Scalar-Tensor equivalent Modified Gravity

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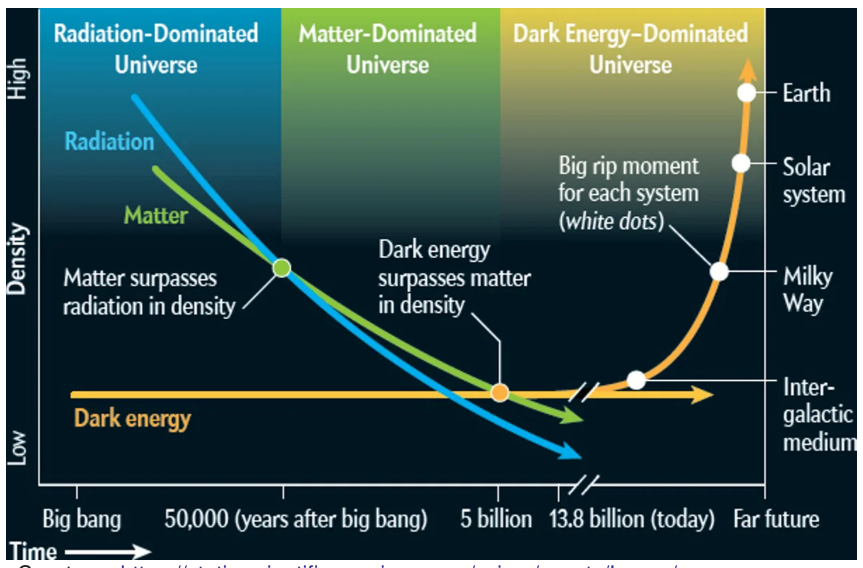
Dynamics of the Cosmos



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An open issue: No consensus on the evolution from the present regime to asymptotic future.

Q. What is the fate of the universe as $t \to \infty$?

To assert this, one needs to do the following:

- Treat the cosmos as a dynamical system.
- Find all the equilibrium states (critical points).
- Identify the stable critical points (attractors).

Interacting Cosmic Dark Sector

Cosmic evolution at low redshifts z is dominated by both dark energy (DE) and dark matter (DM).

- The DE however prevails in the future ($z < 0$).
- In principle, DE and DM may be interacting.
- Such interactions, although allowed in General Relativity (GR), have their forms specified naturally in Scalar-Tensor (ST) theories.

Dark Universe in Modified Gravity

DE, DM (and interactions thereof) perceivable as geometric artifacts in Modified Gravity (MG), rather than sourced by exotic fields (as in GR).

- Most MG formulations have ST equivalents.
- Feature: Non-minimal scalar (ϕ) coupling with the metric in the (original) Jordan frame (JF).
- A conformal transformation to Einstein frame (EF) lifts the non-minimal coupling, in turn inducing ϕ-dependence of matter Lagrangian.
- A DE-DM interaction is hence evident in the Einstein frame, once ϕ is taken to source DE.
- A power-law potential $\sim \phi^{2(2-\alpha)}$ in JF corresponds to the following potential in EF:

$$\boxed{U(\varphi) = \Lambda\, \epsilon^{-2\alpha n\kappa\varphi}}\,, \quad \text{where} \quad \varphi = \frac{\ln[\kappa\phi]}{n\kappa}$$

with $\kappa = \sqrt{8\pi G}$ and (α, n, Λ) are constants.

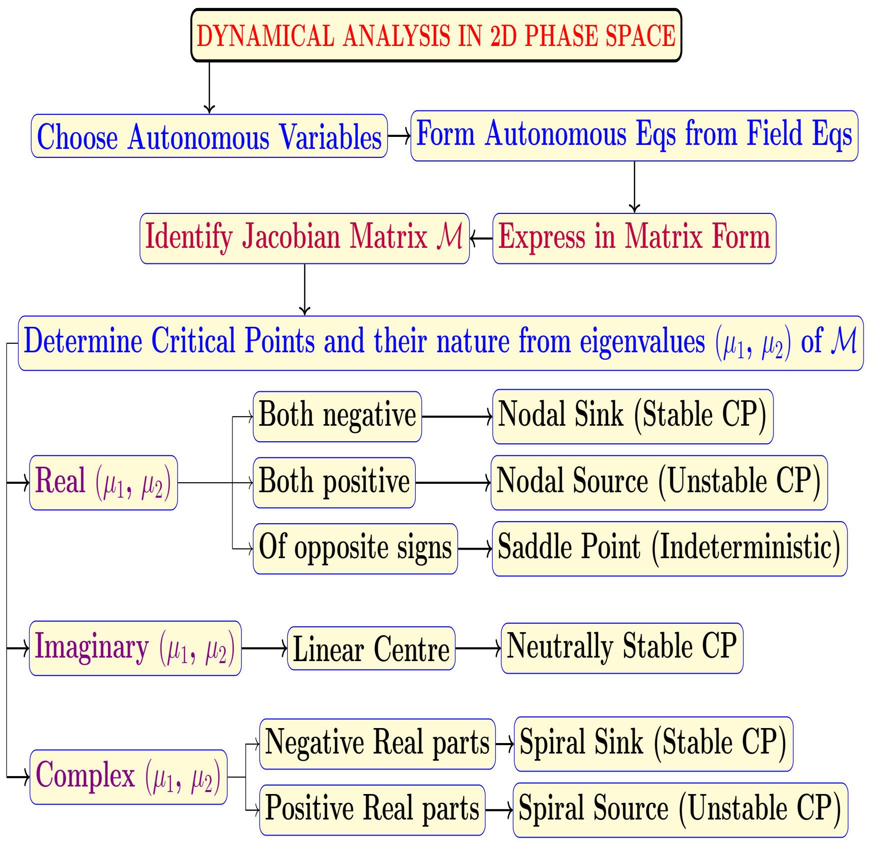
- The constant n specifies φ-matter coupling (or the effective DE-DM interaction).
- Friedmann equations:

$$\boxed{\begin{aligned} H^2 &\equiv \left(\frac{\dot a}{a}\right)^2 = \frac{\kappa^2}{3}\left[\rho_m + \frac{\dot\varphi^2}{2} + U(\varphi)\right] \\ \dot H &\equiv \frac{\ddot a}{a} - \left(\frac{\dot a}{a}\right)^2 = -\frac{\kappa^2}{2}\left(\rho_m + \dot\varphi^2\right) \end{aligned}}\,,$$

where $a(t)$: cosmic scale factor, ρ_m : total (dust-like) matter density, and dot $(\cdot) \equiv d/dt$.

- Due to the φ-coupling $\rho_m \sim a^{-3} \epsilon^{-2\alpha n\kappa\varphi(a)}$.

Dynamical Analysis Methodology



Autonomous System Equations

Autn. variables: $X := \dfrac{\kappa\dot\varphi}{\sqrt{6}H}$, $Y := \dfrac{\kappa\sqrt{U}}{\sqrt{3}H}$.

Autn. eqs and constraint from Friedmann eqs:

$$2X_{,N} = \left(\Omega_m + 2\alpha Y^2\right) n\sqrt{6} - 3\left(\Omega_m + 2Y^2\right) X$$

$$2Y_{,N} = \left[3\left(\Omega_m + 2X^2\right) - 2(2-\alpha)\sqrt{6}X\right] Y\,,$$

$$X^2 + Y^2 = 1 - \Omega_m\,,$$

where $N = \ln a$ and $\Omega_m = \kappa^2\rho_m/(3H^2)$.

Location (X_c, Y_c) of each critical point (CP) is a solution of $\boxed{X_{,N}|_{(X_c,Y_c)} = 0\,,\; Y_{,N}|_{(X_c,Y_c)} = 0}$.

Perturbations $(\delta X, \delta Y)$ about a CP (X_c, Y_c) satisfy the eigenvalue equation

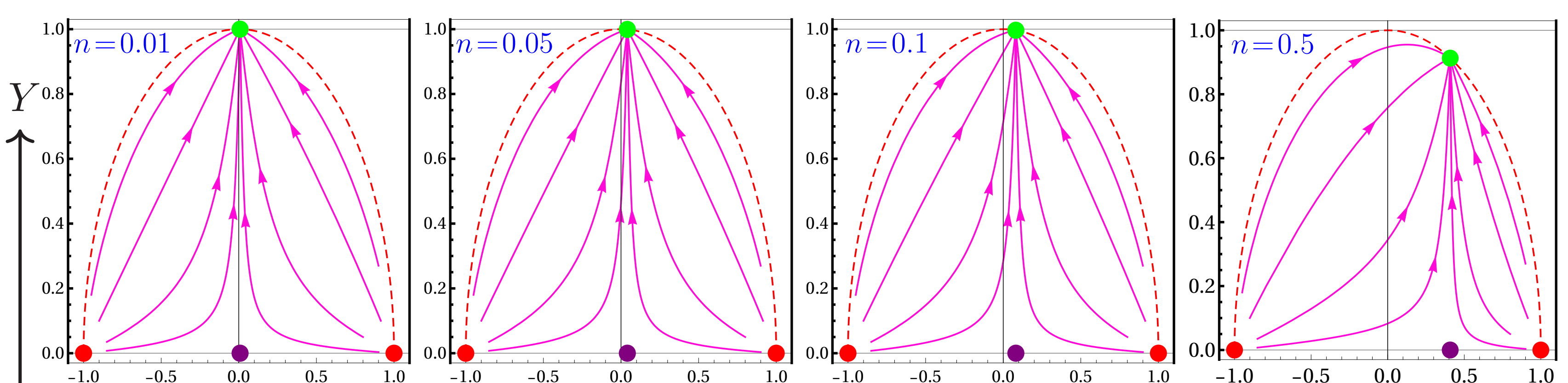
$$\frac{d}{dN}\begin{pmatrix}\delta X\\ \delta Y\end{pmatrix} = \mathcal{M}\begin{pmatrix}\delta X\\ \delta Y\end{pmatrix}$$

where $\mathcal{M}$ is the (2×2) Jacobian matrix whose eigenvalues determine the nature of the CP.

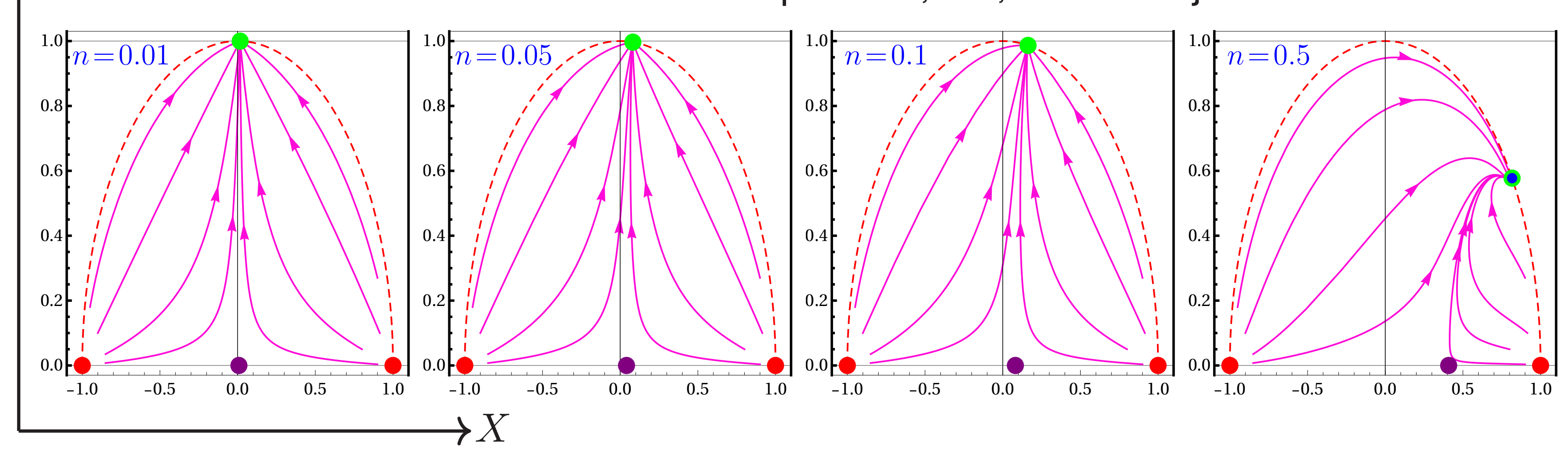
Results of the Dynamical System Analysis

CP	Location of CP (X_c, Y_c)	Domain of n for the existence of CP	Domain of n for the relevance of CP	Value of Ω_m at CP
1	$(-1, 0)$	$(-\infty,\infty)\,,\ \forall\alpha$	$(-\infty,\infty)\,,\ \forall\alpha$	0
2	$(1, 0)$	$(-\infty,\infty)\,,\ \forall\alpha$	$(-\infty,\infty)\,,\ \forall\alpha$	0
3	$\left(\sqrt{\frac{2}{3}}\,n,\ 0\right)$	$(-\infty,\infty)\,,\ \forall\alpha$	$\left[-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right],\ \forall\alpha$	$1 - \frac{2n^2}{3}$
4	$\left(\sqrt{\frac{2}{3}}\,n\alpha,\ \pm\sqrt{1 - \frac{2n^2\alpha^2}{3}}\right)$	$(-\infty,\infty)\,,\quad \alpha=0$ $\left[-\frac{\sqrt{3}}{\sqrt{2}\alpha}, \frac{\sqrt{3}}{\sqrt{2}\alpha}\right],\ \alpha\geq 1$	$(-\infty,\infty)\,,\quad \alpha=0$ $\left[-\frac{\sqrt{3}}{\sqrt{2}\alpha}, \frac{\sqrt{3}}{\sqrt{2}\alpha}\right],\ \alpha\geq 1$	0
5	$\left(\frac{\sqrt{3/2}}{(2\alpha-1)n},\ \pm\frac{\sqrt{3-2(2\alpha-1)n^2}}{\sqrt{2}\,(2\alpha-1)\,n}\right)$	$\left[-\frac{\sqrt{3/2}}{\sqrt{2\alpha-1}}, 0\right) \cup \left(0, \frac{\sqrt{3/2}}{\sqrt{2\alpha-1}}\right],\ \alpha\geq 1$	$\left[-\frac{\sqrt{3/2}}{\sqrt{2\alpha-1}}, -\frac{\sqrt{3/(2\alpha)}}{\sqrt{2\alpha-1}}\right] \cup \left[\frac{\sqrt{3/(2\alpha)}}{\sqrt{2\alpha-1}}, \frac{\sqrt{3/2}}{\sqrt{2\alpha-1}}\right],\ \alpha\geq 1$	$\frac{2\alpha(2\alpha-1)n^2-3}{(2\alpha-1)^2 n^2}$

Numerical Solutions of Autonomous equations, i.e., Phase trajectories for $\alpha = 1$



Numerical Solutions of Autonomous equations, i.e., Phase trajectories for $\alpha = 2$



– Thick dots represent the CPs, and the arrows on the trajectories mark the time-evolution direction.
– Regions inside the dashed boundaries are the ones that support realistic cosmologies ($\Omega_m \geq 0$).
– Stable CPs (or the attractors) for the sub-cases are marked by green dots in respective panels.
– The stable CP is a nodal sink, except for $\alpha = 2, n \geq 0.5$ – spiral sink (see the blue in green dot).

References

[1] E.J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod. Phys. **D 15** (2006) 1753.

[2] S. Sur and A.S. Bhatia, JCAP **07** 039.

[3] A.S. Bhatia and S. Sur, Int J. Mod. Phys **D 26** (2017) 1750149.

[4] M.K. Sharma and S. Sur, Phys. Dark Univ. **40** (2023) 101192.

[5] S. Sur, R. Kapoor, D. Kuniyal and A. Verma, in preparation.

Conclusions

The configuration (or the stable point) to which the universe would transpire to asymptotically, in various MG cases, is thus ascertained by a rigorous analysis in which the cosmos is taken, for simplicity, to be a two-component dynamical system, viz., of (visible+dark) matter and DE.