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Uncertainty Relations in Non-Hermitian Systems

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INTRODUCTION & GOALS

- In non-Hermitian dynamics (open systems / gain–loss), standard means and variances can be inconsistent if the usual inner product is not physically appropriate.
- For pseudo-Hermitian Hamiltonians (including PT symmetry), a metric S is required to define expectation values, variances, and uncertainty relations consistently.
- **Main challenge:** S can become indefinite (broken phase) or singular (exceptional points), motivating a unified framework.
- **Goal:** Construct suitable metrics in each spectral regime (exact phase, broken phase, EPs) within a **Krein-space approach**, and derive a Heisenberg–Robertson-type inequality valid across all regimes.
- **Key:** changing the inner product forces a consistent change of representation: the adjoint changes, and with it the effective form of operators and the representation of states, if physical matrix elements are to remain invariant.

Framework: positive metric, observables, and evolution

Pseudo-Hermiticity: $H^\dagger S = SH, \quad S = S^\dagger$

Indefinite form: $[x, y]_S := \langle x|S|y\rangle, \quad A^\sharp := S^{-1}A^\dagger S$

Positive metric from S ("flip signs"):

$$S = PDP^{-1}, \quad D = \mathrm{diag}(\mu_j), \quad \mu_j \in \mathbb{R}, \qquad S_K := P|D|P^{-1} = |S| > 0,$$

$$\langle x|y\rangle_{S_K} := \langle x|S_K|y\rangle$$

Factorization & operator transport:

$S_K = \gamma^\dagger\gamma \quad \Rightarrow \quad \langle x|y\rangle_{S_K} := \langle x|S_K|y\rangle = \langle \gamma x|\gamma y\rangle$

For canonical $\hat{o} = \hat{o}^\dagger$, let $\hat{O} := \gamma^{-1}\hat{o}\,\gamma$

$$\langle\cdot\,|\,\hat{O}\,|\,\cdot\rangle_S = \langle\cdot\,|\,S\hat{O}\,|\,\cdot\rangle = \langle\cdot\,|\,\gamma^\dagger\gamma\,\hat{O}\,|\,\cdot\rangle = \langle\cdot\,|\,\gamma^\dagger\hat{o}\,\gamma\,|\,\cdot\rangle = \langle\,\gamma\,\cdot\,|\,\hat{o}\,|\,\gamma\cdot\rangle$$

Time evolution (normalized in S_K):

$$|I(t)\rangle = \mathcal{N}(t)e^{-iHt/\hbar}|I(0)\rangle, \qquad \langle I(t)|I(t)\rangle_{S_K} = 1$$

$$\langle\hat{o}(t)\rangle = \langle I(t)\,|\,\hat{O}\,|\,I(t)\rangle_S = \langle\gamma\, I(t)\,|\,\hat{o}\,|\,\gamma\, I(t)\rangle$$

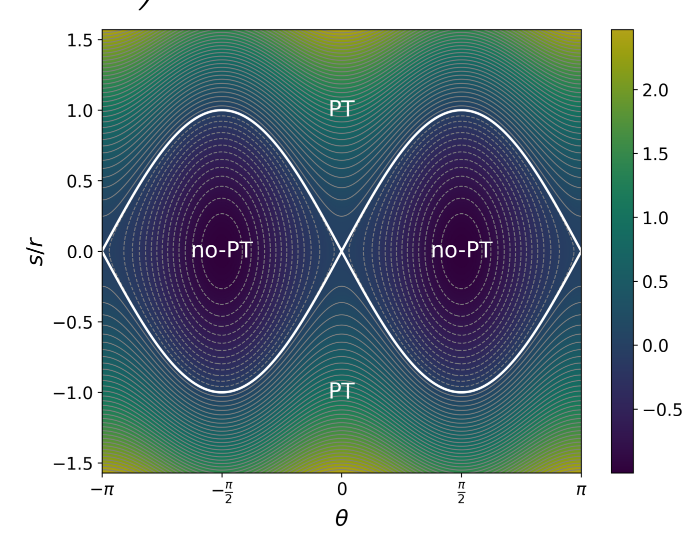
Unified prescription across regimes (overview)

Regime	Spectrum	Metric / inner product
Exact (unbroken)	Real, diagonalizable	$S > 0$ (pseudo-Hermitian metric)
Broken	Complex-conjugate pairs	Krein construction S_K from indefinite S
Exceptional points (EP)	Jordan blocks	Jordan-metric S_J + positive structure

- **The metric is not optional:** it fixes physical means/variances/uncertainty bounds.

MODEL

$$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix} \qquad d = (s/r)^2(1-\eta^2) \qquad \eta = \frac{r}{s}\sin\theta$$



Phase map from $d(s/r, \theta)$: unbroken ($\eta^2 < 1$), broken ($\eta^2 > 1$), EP ($\eta^2 = 1$).

Main inequality (metric form, all regimes)

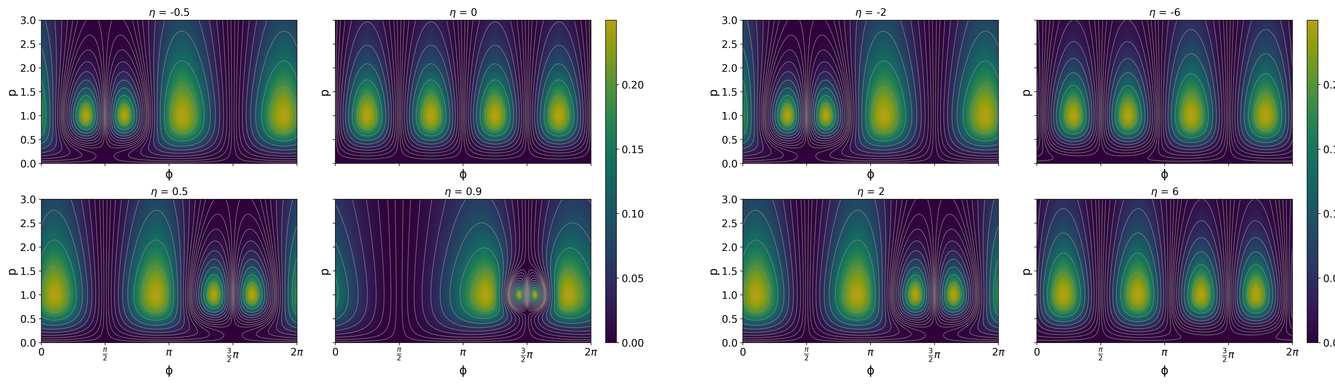
For two non-commuting observables A, B,

$$UR(A, B) := \Delta^2_{S_K}A\,\Delta^2_{S_K}B - \frac{1}{4}|\langle[A, B]\rangle_{SK}|^2 \geq 0$$

- **Same inequality, same look:** the regime-dependence is only in the choice of metric (unbroken: $S > 0$; broken: Krein metric S_K; EP: Jordan-based S_J).

RESULTS & DISCUSSION

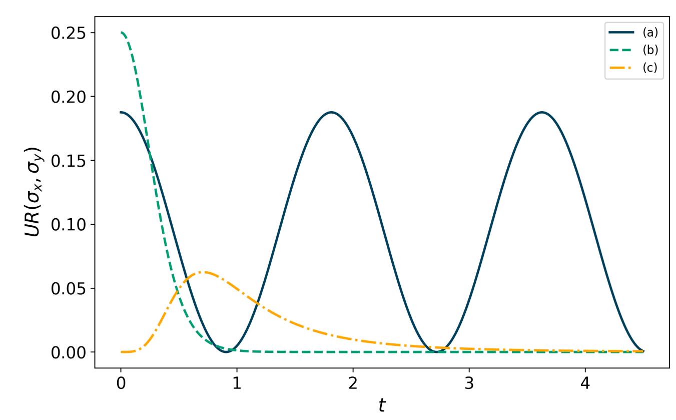
Initial state $|\varphi_0\rangle = (1+p^2)^{-1/2}(|0\rangle + pe^{i\phi}|1\rangle)$



Contour plots of $UR(\sigma_x, \sigma_y)$ in the (ϕ, p) plane at $t = 0$ for different η: left, PT-symmetric region $\eta^2 < 1$; right, broken-symmetry region $\eta^2 > 1$.

Dynamics of uncertainty

- **Unbroken phase:** UR(t) oscillates as a function of time t.
- **Broken phase & EP:** UR(t) relaxes to a **minimum-uncertainty** steady state.



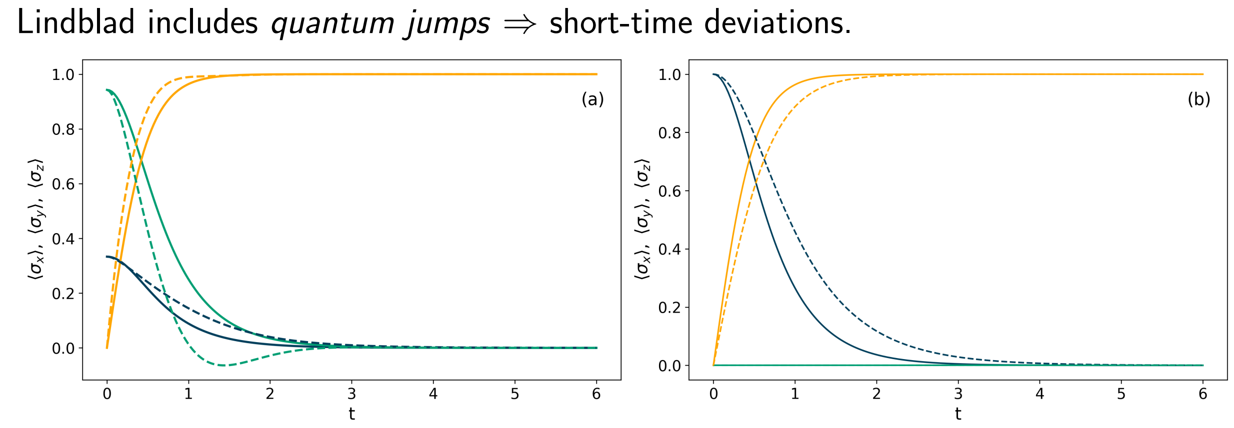
(Animation- Time evolution)

Time evolution of $UR(\sigma_x, \sigma_y)$: (a) exact PT phase ($\eta = 1/2,\ s = 1$), (b) broken PT phase ($\eta = \sqrt{2},\ s = 1$), (c) EP ($\eta = 1,\ s = 1$). Initial state $(\phi, p) = (\pi, 1)$.

Steady-state predictions match the Lindblad master-equation description (differences only at early times: jumps / postselection).

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[h, \rho] + \sum_k \gamma_k \left(L_k\rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}\right).$$

- Hamiltonian: $h = r\cos\theta\,\mathbb{I} + s\,\sigma_x$.
- Collapse operators: $L_\pm = (\sigma_x \pm i\sigma_y)/2$, $\gamma_k = \sqrt{|r\sin\theta|}$.
- Non-Hermitian evolution ∼ effective/no-jump (postselected) trajectory.
- Lindblad includes *quantum jumps* ⇒ short-time deviations.



Behavior of the mean values of the components of spin, $\langle\sigma_i\rangle_{S_K}$, as a function of time (blue, green and yellow dashed lines correspond to $\langle\sigma_x\rangle_{S_K}$, $\langle\sigma_y\rangle_{S_K}$, $\langle\sigma_z\rangle_{S_K}$ respectively). Solid lines show the effective (no-jump) evolution.

CONCLUSION

- The metric determines physically meaningful expectation values, variances and uncertainty relations in exact, broken and EP regimes within a single prescription.
- In pseudo-hermitian dynamics, Krein-space tools show oscillations in the exact phase; relaxation to stationary minimum-uncertainty states in the broken phase and at EPs.
- Lindblad benchmarking supports the steady-state predictions. At early stages of time evolution, the differences support the role of postselection.

OUTLOOK & REFERENCES

Establish a constructive equivalence between the broken-PT Krein-space formulation and the metric-operator canonical-inner-product approach, including an explicit dictionary for metrics, adjoints, observables, and states.

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