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Operator-based Comparative Study of Anomalous Thermal Diffusion

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INTRODUCTION & AIM

Heat transfer in solids has been attracting the attention of researchers for a long time. The primary objective of these studies has been to establish a relationship between heat flux and the temperature gradient. The first classical model that describes this relationship is Fourier's law, originally introduced by Joseph Fourier based on experimental observations. While effective for macro-scale systems, this classical model is not applicable in certain scenarios, such as heat transfer in non-homogeneous materials and complex media (porous materials, polymers, biological tissues), under high-intensity heat sources like laser pulses, or at very small temporal and spatial scales. In these cases, heat transport exhibits anomalous behavior driven by memory effects, nonlocal interactions, and structural heterogeneity. To address these limitations, non-Fourier heat transfer theories have emerged. In recent decades, it has become widely recognized that partial differential equations of non-integer order (fractional differential equations, FDEs) provide an effective mathematical framework for describing anomalous transport phenomena. The concept of fractional calculus originated in 1695, following the correspondence between L'Hopital and Leibniz regarding the interpretation of non-integer derivatives $D^n f(t)=d^n f(t)/dt^n$ (*n* is a non-integer.) Since then, numerous prominent scientists have contributed to the development of this field (see Figure 1 for an overview). An important advantage of this formalism is its ability to incorporate boundary conditions and external forces in a relatively straightforward manner. Moreover, it enables the use of well-established tools from mathematical physics and statistics to derive expressions that describe complex systems exhibiting anomalous behavior. Over the years, several definitions of fractional derivatives have been introduced, including Caputo, Caputo-Fabrizio, and Atangana-Baleanu fractional derivatives. These different definitions arise from the aim to preserve various properties of the classical integer-order derivative. An important feature of such operators is that memory effects are implicitly incorporated in their formulations.

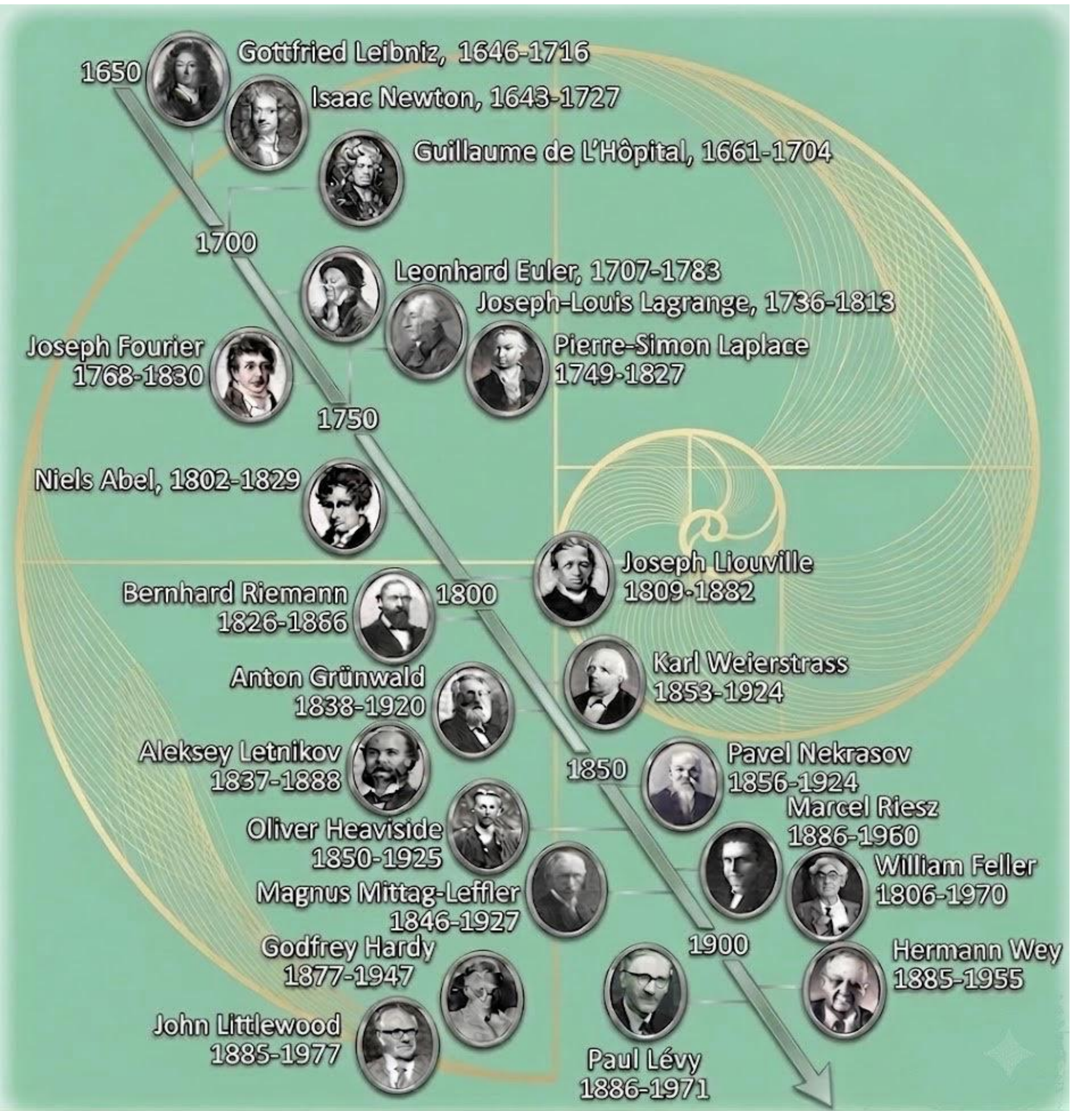

Figure 1. Timeline highlighting key scientists in the development of fractional calculus. ***This figure was created with the AI tool Gemini, based on and adapted from the illustration in "Some Pioneers of the Application of Fractional Calculus," 2013, DOI:10.1115/DETC2013-12471.***

The primary goal of this research is to investigate how different fractional operators influence the amplitude and phase characteristics of temperature variations in surface-illuminated samples. This study specifically examines the differences in predicting subdiffusive heat propagation within 2D graphene oxide structures subjected to laser pulse irradiation. To capture the non-Fourier thermal behavior, three distinct time-fractional operators are employed: Caputo - singular kernel, Caputo-Fabrizio (CF) – non-singular, exponential kernel, and Atangana-Baleanu (ABC) – non-singular, Mittag-Leffler kernel. The models are defined under Neumann boundary conditions and solved using Laplace transform method.

METHOD

We consider a one-dimensional heat conduction model for a thin film, subjected to a laser pulse at the surface (x=0). To analyze the impact of different kernels on subdiffusive propagation, three distinct operators are incorporated into the governing time-fractional heat conduction equation:

$$ {}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^{\alpha}} \frac{df(\tau)}{d\tau} d\tau, $$

$$ {}^{CF}D_t^{\alpha} f(t) = \frac{1}{1-\alpha} \int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right) \frac{df(\tau)}{d\tau} d\tau, $$

$$ {}^{ABC}D_t^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right) \frac{df(\tau)}{d\tau} d\tau. $$

The system is subject to Neumann boundary conditions, representing the heat flux induced by laser irradiation at the front surface and insulation at the rear surface. The resulting fractional differential equations are solved using the Laplace transform method, transforming the time-dependent equations into the s-domain.

RESULTS & DISCUSSION

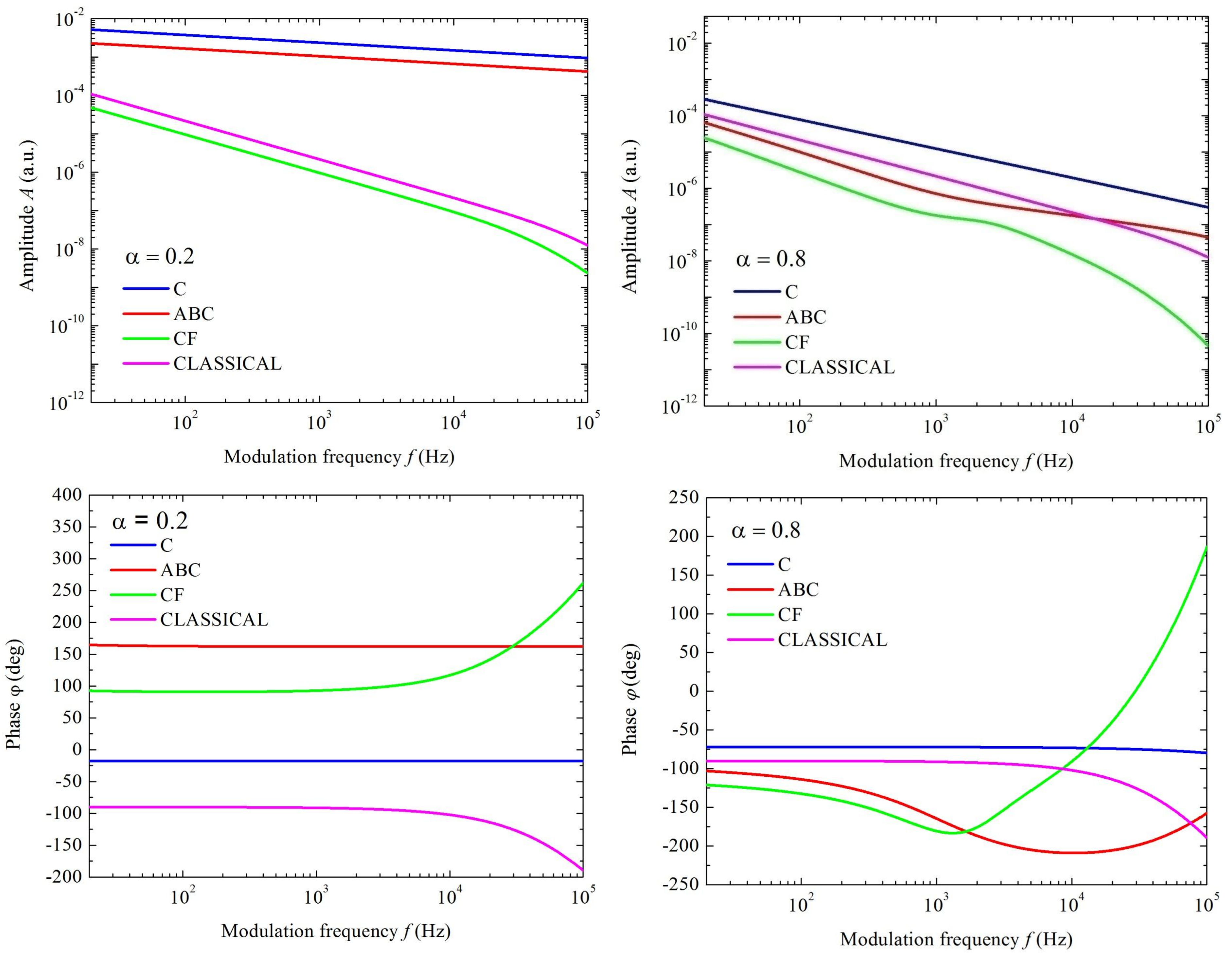

Figure 2. Amplitude and phase characteristics: left panels corespond to fractional order 0.2 and right panels to 0.8.

The final solutions are analyzed in the acoustic frequency range to determine amplitude response (how the magnitude of temperature fluctuations decays with depth and frequency) and phase characteristics (the time lag in thermal wave propagation).

The comparative analysis in the frequency domain reveals distinct transport signatures for each operator.

For higher values of the fractional order, the CF and ABC operators exhibit a similar trend at lower frequencies. Conversely, for lower fractional orders, the classical and Caputo models show similar phase characteristics at low frequencies, while significant separation occurs at higher frequency ranges.

The mathematical consistency of the fractional models is validated by the limit case fractional order (fractional ored 1), where the Caputo, CF, and ABC fractional operators converge exactly to the classical model predictions.

CONCLUSION

This study demonstrates that fractional operators are essential for capturing transport patterns that classical integer-order models fail to describe.

The findings have potential applications in photothermal characterization and imaging of various functional materials and biological tissues.

FUTURE WORK / REFERENCES

Further investigations will examine how sample thickness and intrinsic thermal diffusivity modify the thermal signature within the acoustic frequency range.

References

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