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Structural Optimization of Integer-Order Approximations for Fractional Differential Equations

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INTRODUCTION & AIM

This work addresses the inherent ambiguity in separating a fractional system's dynamics to perform a structural analysis of the FDE. Instead of relying on a single, manually derived representation, this study treats the integer-order component $G(y(x))$ as a flexible, parameterizable, and optimizable polynomial.

Primary Objectives:

- **Optimize Model Architecture:** To introduce a global optimization framework (using Particle Swarm Optimization) that searches the space of possible ODE architectures by parameterizing the polynomial coefficients of $G(y(x))$.
- **Minimize Dual Objectives:** To identify an optimal, low-degree polynomial structure that simultaneously minimizes the solution error (compared to a high-fidelity reference) and the magnitude of the truncated residual series $\Psi_M(t^n)$.
- **Uncover Structural Insights:** To provide a novel analytical tool that reveals deeper fundamental insights into the complex interplay between local, integer-order dynamics and non-local memory effects within fractional systems.

FDE to ODE TRANSFORMATION

1. Initial FDE

$$\left({}^C\mathbf{D}^{(1/n)}\right)^k y(x) = F\left(y(x)\right), \quad 1 \le k < n;$$

$$y(0) = v_0, \left({}^C\mathbf{D}^{(1/n)}\right)^j y(x)|_{x=0} = v_j, \quad j = 1, \dots, k-1.$$

2. Equivalent First-Order ODE

Apply substitution $t = x^{1/n}$ (with $\widehat{y}(t) = y(t^n)$) to reduce the transformed FDE to an equivalent first-order ODE:

$$\frac{\mathrm{d}\widehat{y}}{\mathrm{d}t} = nt^{n-1}\left(G(\widehat{y}(t)) + \Psi(t^n)\right) + \sum_{j=1}^{n-1} \frac{jv_j}{\Gamma(1+\frac{j}{n})} t^{j-1}$$

3. Approximate ODE for Optimization

Truncate $\Psi(t^n)$ for numerical computation ($\Psi(t^n) \approx \Psi_M(t^n)$), yielding the final approximate ODE:

$$\frac{\mathrm{d}\widetilde{y}}{\mathrm{d}t} \approx nt^{n-1}\left(G(\widetilde{y}(t)) + \Psi_M(t^n)\right) + \sum_{j=1}^{n-1} \frac{jv_j}{\Gamma(1+\frac{j}{n})} t^{j-1}$$

OPTIMIZATION ALGORITHM

1. Parameterize $G(y)$

Define $G(y)$ as a polynomial whose coefficients $\mathbf{b}$ are the target for optimization:

$$G(y;\mathbf{b}) = \sum_{p=0}^{N_G} b_p y^p$$

2. Establish Normalization Baseline

Define a baseline architecture $\mathbf{b}_{\mathrm{baseline}}$, using the original FDE's structure. Calculate its performance metrics to serve as reference values for normalization:

$$\mathrm{RMSE}_{\mathrm{ref}} = \mathrm{RMSE}(\mathbf{b}_{\mathrm{baseline}}); \quad \|\Psi_M\|_{\mathrm{ref}} = \|\Psi_M(\mathbf{b}_{\mathrm{baseline}})\|_2$$

3. Iterative Optimization Loop (PSO)

A. Generate Candidate & Solve: For a candidate vector $\mathbf{b}$, construct and solve the corresponding approximate ODE to get the solution $\widetilde{y}(t;\mathbf{b})$.

B. Evaluate Fitness: Calculate the solution accuracy (RMSE) and model simplicity ($\|\Psi_M\|_2$) and combine them into a single normalized objective function:

$$\mathrm{Minimize}_{\mathbf{b}} \quad T_{\mathrm{norm}}(\mathbf{b}) = \kappa_1 \frac{\mathrm{RMSE}(\mathbf{b})}{\mathrm{RMSE}_{\mathrm{ref}}} + \kappa_2 \frac{\|\Psi_M(\mathbf{b})\|_2}{\|\Psi_{M,\mathrm{ref}}\|_2}$$

4. Optimal Architecture

The optimization yields the coefficient vector $\mathbf{b}^*$, which defines the optimal architecture $G(y;\mathbf{b}^*)$.

CONCLUSION

By balancing solution accuracy with structural simplicity, the proposed optimization framework identifies parsimonious integer-order representations of fractional differential equations, providing a novel analytical tool to uncover the fundamental interplay between local dynamics and non-local memory effects.

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