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Comparative Numerical Analysis of Bessel Integrals Arising in Space-Fractional Diffusion

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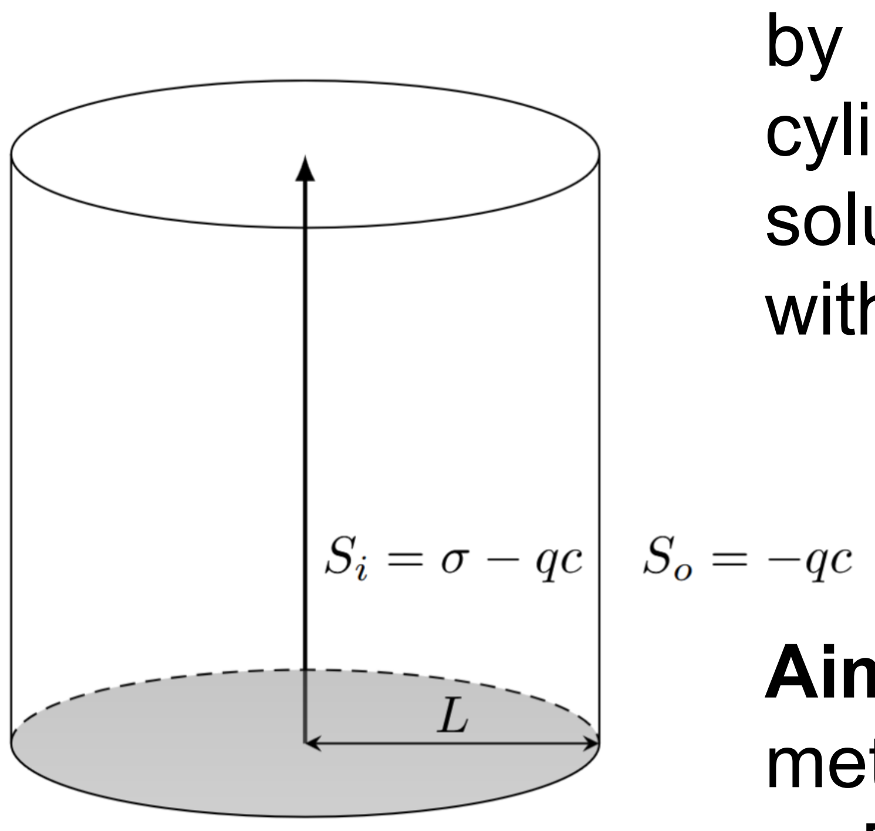
INTRODUCTION & AIM



Diffusion in biological tissues (e.g., brain) deviates from Fick's laws due to microscopic heterogeneity – modeled by space-fractional Riesz Laplacian. In cylindrical geometry, the steady-state solution reduces to a Hankel transform with oscillatory Bessel integrals:

$$c(r) = \sigma L \int_0^\infty \frac{J_0(\rho r) J_1(\rho L)}{\rho^{2\alpha} + q}\, d\rho$$

Aim: Compare three quadrature methods:

- Double-Exponential with Wynn's acceleration
- Ogata method (sinc-based for oscillatory kernels)
- Sinc integration rule (Denich–Novati)

METHOD

•**Model:** Space-fractional reaction-diffusion:

$$-(-\Delta)^\alpha c + \sigma - qc = 0$$

•**Hankel transform** gives the integral above.

•**Three numerical methods:**

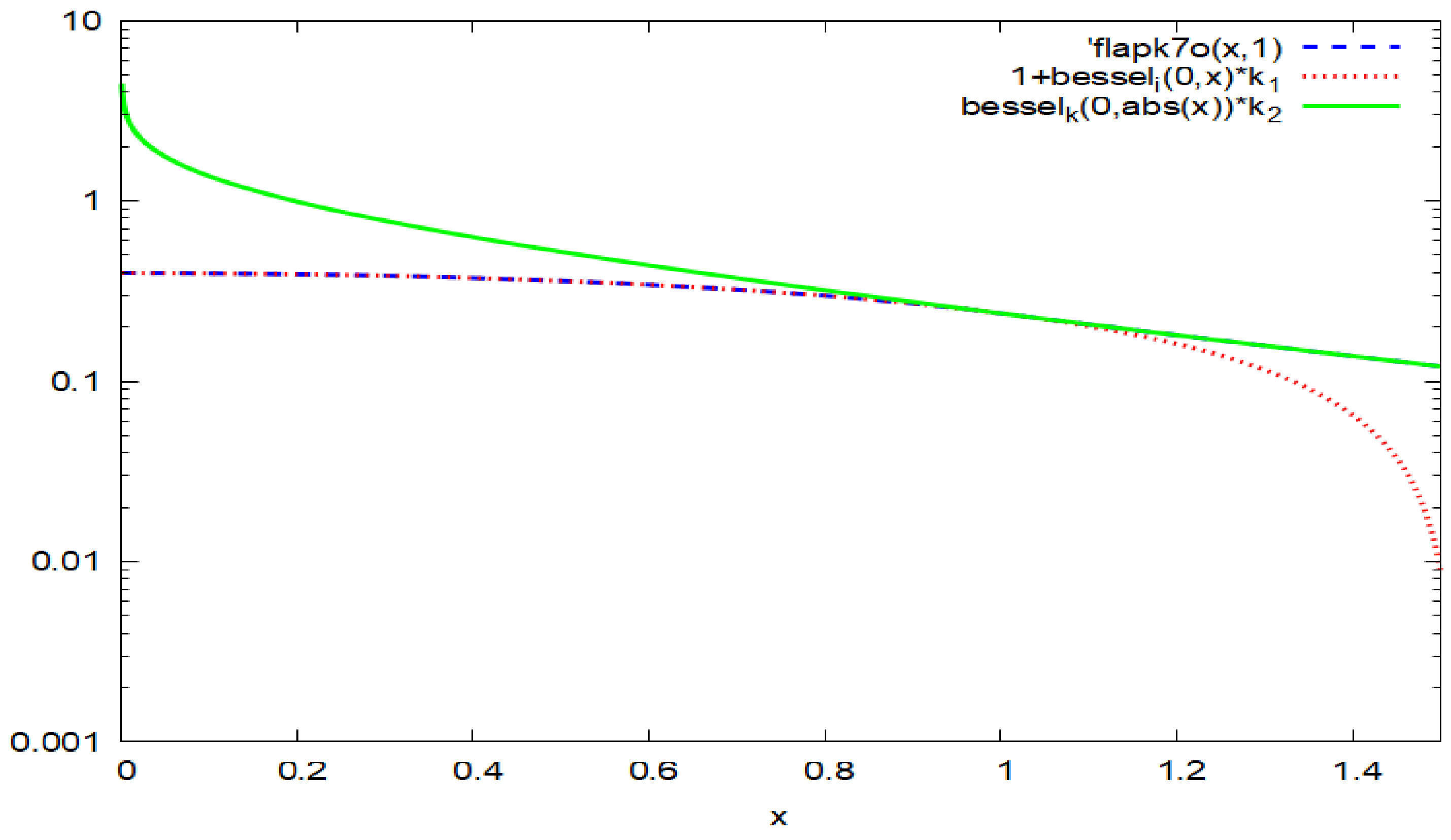
- DE (Double-Exponential) + Wynn's ϵ acceleration (Ooura & Mori)
- Ogata method (sinc-based for oscillatory kernels)
- Sinc rule (Denich & Novati, 2024) with transform

$$\rho = \frac{T}{r} \phi(\xi - p)$$

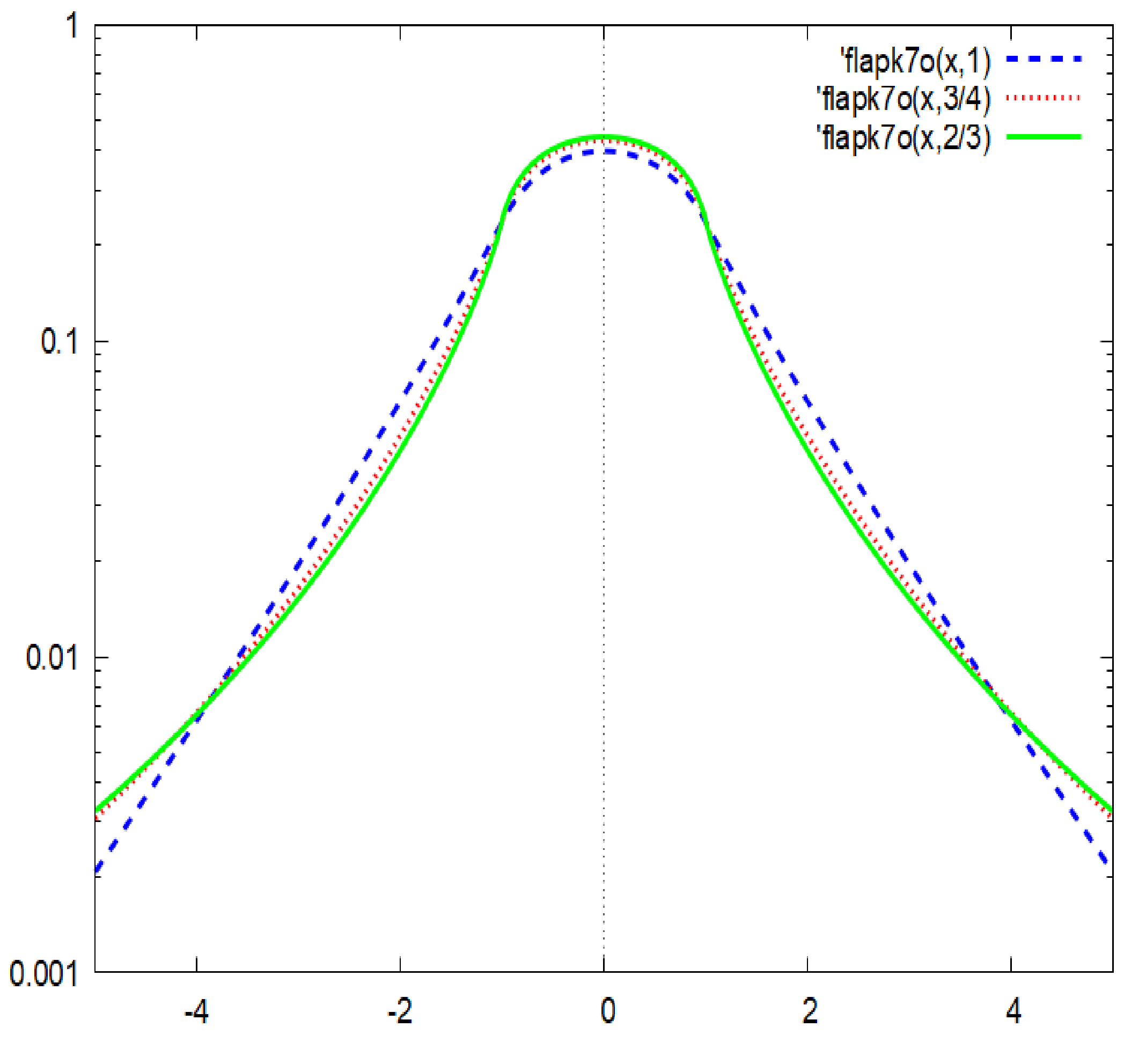
DE (Double-Exponential)	$\rho = \exp\left(\frac{\pi}{2} \sinh t\right)$	Wynn's ϵover Bessel zero intervals
Ogata	Sinc-based double-exponential for oscillatory kernels	-
Sinc (Denich–Novati)	$\rho = \frac{T}{r}\phi(\xi - p)$, $\phi(\xi) = \frac{\xi}{1-e^{-\xi}}$	Single-exponential transform

RESULTS & DISCUSSION

Validation (α=1α=1) – DE recovers analytical Bessel solution. Integer-order inner/outer solutions.



Fractional order effect – Lower α gives heavy-tailed profiles, slower decay. Full solution for α=1,3/4,2/3α=1,3/4,2/3:



α=3/4

r	c(r)
1	0.15613
2	0.03488
3	0.01020
4	0.00328

Method	Relative error	Speed	Convergence
DE + Wynn	< 1e-10	moderate	exponential
Ogata	< 1e-10	fast	double-exp
Sinc (Denich)	< 1e-12	fastest	double-exp

CONCLUSION

- DE, Ogata, and Sinc methods yield identical results to machine precision.
- Sinc rule offers best balance of speed, accuracy, and simplicity.
- Reliable numerical evaluation enables parameter estimation from experimental data (e.g., neural electrodes).

FUTURE WORK / REFERENCES

Future work: Application to experimental impedance data

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•Prodanov D. (2022) *First-order reaction-diffusion with s-f diffusion*
•Denich & Novati (2024) *J. Sci. Comput.* 100:23.
•Ooura & Mori (1991) *J. Comput. Appl. Math.* 38:353–360.
•Ogata (2005) *J. Comput. Appl. Math.* 174:379–396.