

Maximum Likelihood Estimation and Properties of the Transmuted Exponential–Weibull–Exponential Distribution and its Comparison with Related Probability Distributions

Nnaemeka Martin Eze^{1*} and Oluchukwu Chukwuemeka Asogwa²

¹ Department of Statistics, University of Nigeria, Nsukka.

² Department of Mathematics and Statistics, Alex Ekweme Federal University Ndufu-Alike, Nigeria.

*Corresponding author: nnaemeka.eze@unn.edu.ng

INTRODUCTION & AIM

Introduction

Lifetime data are well described using lifetime distributions, but not all lifetime distributions are appropriate for modeling some special lifetime data. Instead of transposing the lifetime data, it is better to extend the existing models, that is, to use statistical or mathematical approach to develop a new model that will best fit the data. Some of the most widely applicable lifetime distributions includes Weibull, exponential, gamma, normal, lognormal, and Lindley distributions. Many authors introduced different techniques to extend or generate statistical distributions for modeling data for a better fit. The formation of a new distribution from the existing one is mostly done by inducing additional parameters to the baseline model.

Some methods of developing new distributions include; Transmuted-G by Shaw and Buckley (2007), Transmuted Exponentiated Generalized-G (TEG-G) by Yousof et al. (2015), Transmuted Exponential-G by Mohammed and Ugwuowo (2020). In literature, numerous families of Transmuted-G generators by Shaw and Buckley (2007) have been studied, and proven that these distributions give better fits than the baseline distributions. Some of these distributions include; Transmuted exponentiated gamma by Hussian (2014), Transmuted Gompertz by Abdul-Moniem and Seham (2015), Transmuted modified Weibull by Khan and King (2013), Transmuted complementary Weibull Geometric by Afify et al. (2014), Transmuted Weibull Lomax by Afify et al. (2015), Transmuted Exponentiated Weibull Geometric by Saboor et al. (2016), Transmuted New Weibull-Pareto by Tahir et al. (2018), Transmuted Exponential-Weibull by Mohammed and Ugwuowo (2020), Transmuted Topp-Leone Weibull by Ibrahim and Yousof (2014) etc. Flexible lifetime distributions play a central role in both the theoretical development and practical application of survival and reliability analysis. Emerging experimental lifetime data in engineering and biological sciences often display skewness, heavy tails, and non-monotonic hazard rates that classical lifetime distributions fail to capture.

Aim of the Study

The aim of this study is to develop a lifetime distribution called Transmuted Exponential-Compound Weibull-Exponential (TE-W-E) which will be used to model lifetime right-skewed datasets for which the existing lifetime distributions cannot capture efficiently well.

Objectives of the Study

The objectives of the study are to: investigate the performances of some of the existing lifetime distributions; (ii) propose a new lifetime distribution called Transmuted Exponential-Weibull-Exponential (TE-W-E) to model lifetime datasets; (iii) validate the efficiency of the proposed lifetime distribution vis-à-vis some of the existing ones on real-life datasets; (iv) estimate the parameters of the proposed distribution using Maximum Likelihood Estimation (MLE).

Significance of the Study

TE-W-E model practical significance arises from its ability to model different types of hazard functions, which makes it suitable across many disciplines such as medicine, engineering, psychology, pharmacy, veterinary medicine, meteorology, environmental sciences etc.

METHOD

The TE-W-E distribution is constructed using the transmuted exponential-G generator proposed by Mohammed and Ugwuowo (2020) to extend the Weibull-Exponential distribution in order to increase its flexibility. The flexibility of the Weibull-Exponential distribution was increased by inducing scale and transmuted parameters. Comprehensive derivations of the mathematical and statistical properties of the proposed TE-W-E distribution such as the probability density, cumulative distribution, ordinary and central moments, moment generating function, mean, variance, quantile function, survival function, hazard function, reversed hazard function, odd function, cumulative hazard function, Renyi entropy and order statistics were analytically derived and are presented. Model parameters were estimated using the maximum likelihood method and the observed Fisher information matrix was derived to establish the asymptotic inference. Monte Carlo simulations were conducted to evaluate estimator performance under different sample sizes in terms of bias, variance, and mean square error.

The cdf of TE-W-E distribution is obtained as:

$$F(x; \tau, \omega, \alpha, \delta) = \left(1 - \left(\frac{\delta}{x^\alpha + \delta}\right)^\tau\right) \left(1 + \omega \left(\frac{\delta}{x^\alpha + \delta}\right)^\tau\right) \quad (1)$$

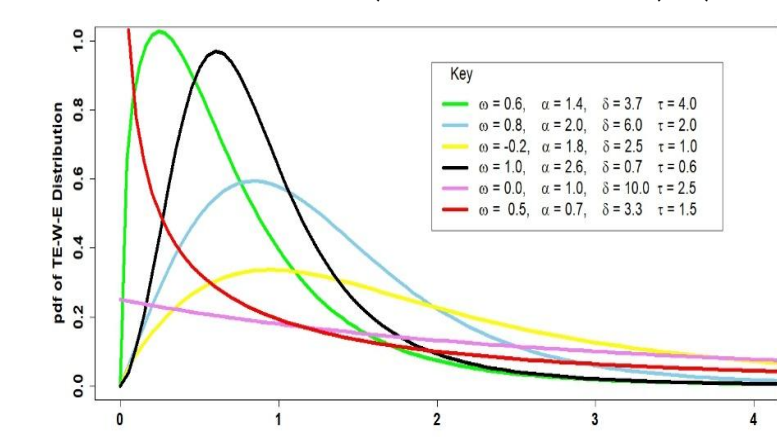


Fig. 1: Plot of pdf for some arbitrary parameter values of TE-W-E distribution

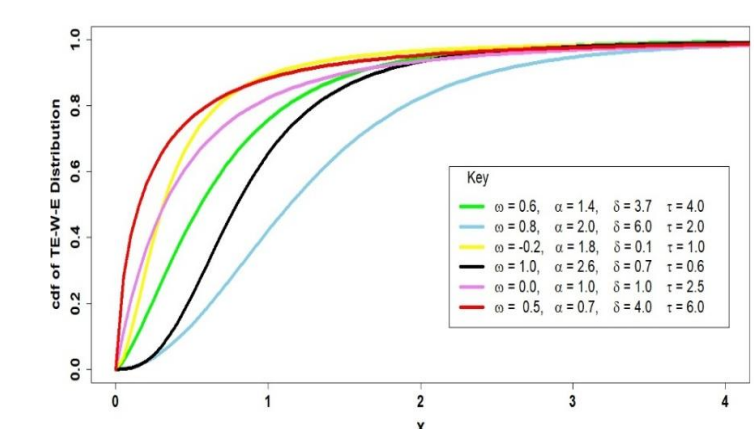


Fig. 2: Plot of cdf for some arbitrary parameter values of TE-W-E distribution

Fig. 1 shows that the pdf of the proposed TE-W-E distribution is positively skewed and thus, will be a good model for modeling positive skewed datasets. The density also showed different degrees of skewness and kurtosis due to changes in the shape parameter, α values.

Fig. 2 shows that the cdf of the proposed TE-W-E distribution is a non-decreasing function. It shows that the line in the cdf plot of the TE-W-E distribution starts from zero and trends upward. It also shows that there is a linear relationship between time and unreliability function, $F(x)$.

Fig. 3 show that the value of the survival function equals 1 at initial time and it decreases as time on x-axis increases and equals zero as time on x-axis turns to be larger. This means that the TE-W-E distribution is appropriate in modeling time or age-dependent events, where survival decreases with time.

Fig. 4 shows that the hazard function of the proposed TE-W-E distribution has J-shaped or bathtub-shaped. The J-shaped or bathtub-shaped of the hazard function of the proposed TE-W-E distribution shows that it is highly flexible to model any right skewed data.

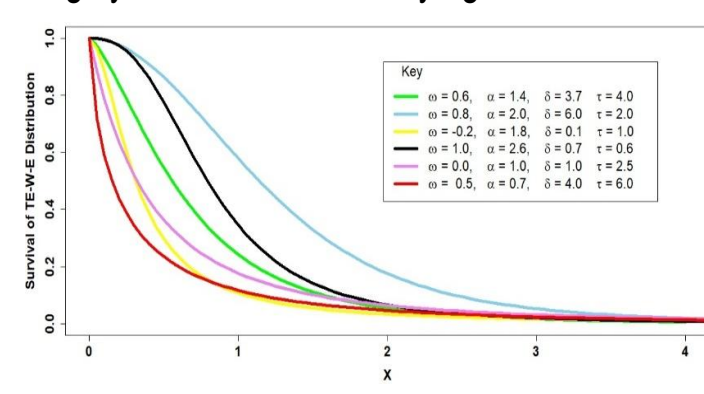


Fig. 3: Plot of survival function for some arbitrary parameter values of TE-W-E distribution

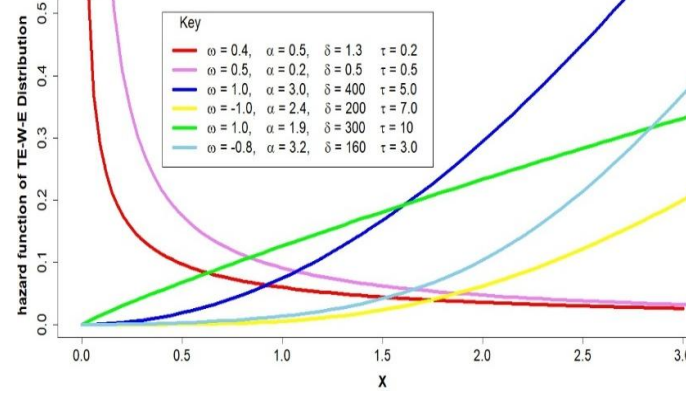


Fig. 4: Plot of survival function for some arbitrary parameter values of TE-W-E distribution

REFERENCES

- Afify, A. Z., Nofal, Z. M., Yousof, H. M., El Gebaly, Y. M., & Butt, N. S. (2015). The transmuted Weibull Lomax distribution: Properties and application. *Journal of Statistics and Operation Research*, 11(1), 135–152.
- Dubey, S. D. (1968). A compound Weibull distribution. *Naval Research Logistics Quarterly*, 15, 179–188.
- Mohammed, A. S., & Ugwuowo, F. I. (2020). A new four-parameter Weibull distribution with application to failure time data. *FUDMA Journal of Sciences*, 4(3), 563–575.
- Mohammed, A. S., & Ugwuowo, F. I. (2021). The transmuted exponential-exponential distribution with application to breast cancer data. *Pakistan Journal of Statistics*, 37(1), 1–18.
- Okorie, I. E. (2020). *Contributions to distribution theory with applications* (Doctoral dissertation, University of Manchester).
- R Core Team. (2024). R: A language and environment for statistical computing (Version 4.4.1). R Foundation.
- Shaw, W. T., & Buckley, I. R. C. (2007). The alchemy of probability distributions.
- Weibull, W. (1951). A statistical distribution function of wide applicability. *Journal of Applied Mechanics*, 18, 293–297.
- Yousof, H. M., Afify, A. Z., Alizadeh, M., Butt, N. S., Hamedani, G. G., & Ali, M. M. (2015). The transmuted exponentiated generalized-G family of distributions. *Pakistan Journal of Statistics and Operation Research*, 11(4), 444–464.

RESULTS & DISCUSSION

Application to a Real Life Dataset

The potentiality of TE-W-E distribution is illustrated in this section through the application of a real-life dataset. The TE-W-E distribution is compared with other competing distributions; Weibull (W) (Weibull, 1951), Weibull-Exponential (W-E) (Dubey, 1968), Weibull-Gamma (W-G) (Dubey, 1968), Kumaraswamy-Weibull (Kum-W) (Cordeiro et al., 2010), Transmuted Weibull Lomax (TWL) (Afify et al., 2015), Generalized Transmuted-Weibull (GT-W) (Nofal et al., 2017), Transmuted New Modified Weibull (TNMWD) (Vardhan & Balaswamy, 2016), Transmuted Lindley (TL) (Merovci, 2013a), Transmuted Exponential-Weibull (TE-W) (Mohammed & Ugwuowo, 2020) and New Exponential-Exponential (NE-E) (Ogunwale et al., 2022). The maximum likelihood estimates of the parameters of the distributions and its standard errors are obtained. Some assessment criteria are used to determine the best model out of the competing models. These criteria include Log-Likelihood (LL), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Hannan-Quinn Information Criteria (HQIC) and Bayesian Information Criterion (BIC). Furthermore, we equally compute other measures such as Anderson-Darling (A*), Cramér-Von Mises (W*) and Kolmogorov-Smirnov (D*) test statistic. The null hypothesis is that data follow a specified distribution. In general, model with the smallest value of the test statistics is considered to be the best fit among the competing models.

Dataset

The second dataset is on the survival times of Guinea Pigs injected with different doses of tubercle bacilli collected by Bjerkedal (1960). Recently, Okorie (2020) fitted the Transmuted Lindley (TL) distribution to this data.

Table 1: Descriptive statistics for Dataset

Dataset	Sample size	Min.	Q1	Median	Mean	Q3	Max.	Variance	Skewness	Kurtosis
Dataset	72	12.00	54.75	70.00	99.82	112.75	376.00	6580.122	1.7590	2.4596

The data was analyzed using R version 4.4.2 software. The skewness and kurtosis were obtained using fBasics package and the parameter estimates were obtained using optim function in R (R Core Team, 2024); and Table 2 presents the computed maximum likelihood estimates and goodness-of-fit measures obtained.

Table 2: Estimated Parameters and Goodness-of-Fit for Dataset

Model	ML Estimates	-LL	AIC	CAIC	HQIC	BIC	W*	A*	D*
TE-W-E	$\tau = 1.3413$ $\omega = -0.9942$ $\alpha = 1.8299$ $\delta = 1800.0$	389.8173	787.6347	779.7541	791.2601	785.4474	0.1009	0.5794	0.09221
W-E	$\alpha = 1.8759$ $\delta = 3000.0$ $\beta = 1.3070$	394.2842	792.5683	788.6263	794.3810	791.4747	0.3241	1.8729	0.1613
W-G	$\gamma = 2.6417$ $\eta = 800.0027$	399.2953	804.5907	798.6789	807.3097	802.9502	0.6892	3.7841	0.21243
TL	$\alpha = 0.0159$ $\lambda = 0.0570$ $a = 1.6802$ $b = 3.9960$	393.2405	790.4809	786.5390	792.2937	789.3873	0.2597	1.4933	0.1084
TWL	$\beta = 1.9864$ $\alpha = 0.1531$ $\lambda = 0.3578$ $\lambda = 0.5997$	393.7529	797.5058	787.6573	802.0376	794.7717	0.3102	1.7848	0.1464
TE-W	$k = 1.5232$ $\gamma = 100.0046$ $\theta = 0.5974$ $\beta = 1.3169$ $\alpha = 0.0100$	395.4364	798.8728	790.9922	802.4982	796.6855	0.3433	1.9626	0.1286
GT-W	$a = 1.2709$ $b = 0.3464$ $\lambda = 0.3164$	395.8325	801.6651	791.8166	806.1968	798.9309	0.4072	2.1845	0.1446
NE-E	$\lambda = 0.1001$ $\beta = 1.3443$	403.4421	808.8843	806.9128	809.7906	808.3374	0.8060	4.4728	0.2116
W	$\delta = 100.7926$	397.6633	799.327	795.385	801.139	798.233	0.3512	2.372	0.1424

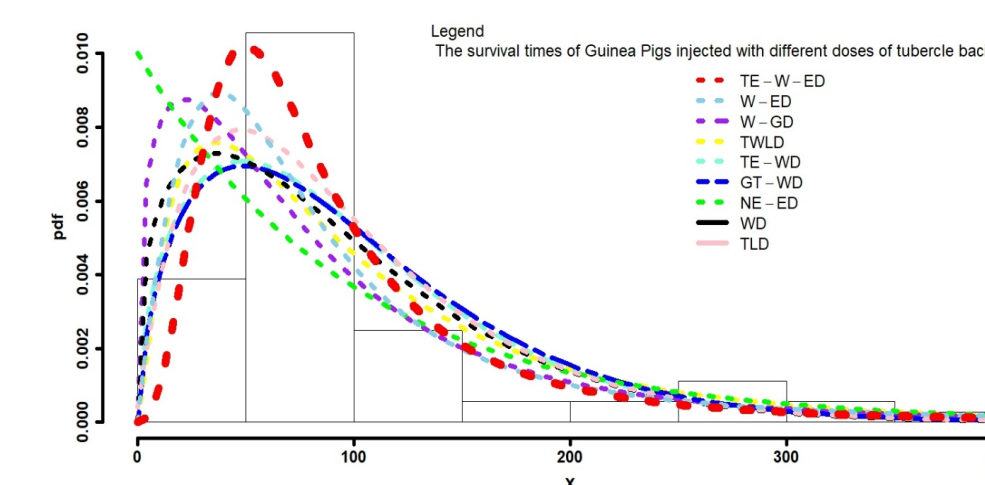


Fig. 5: Plot of the estimated densities for the fitted models to dataset

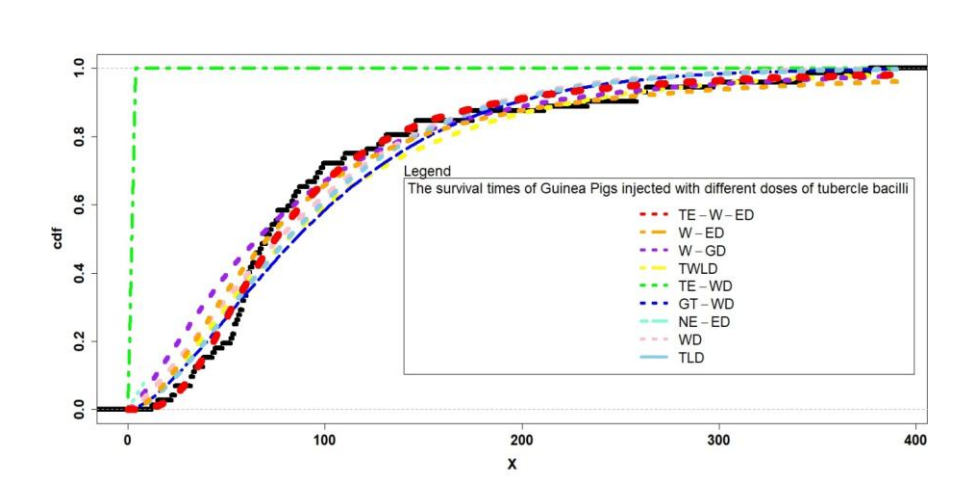


Fig. 6: Plot of the cdf of the fitted models and empirical cdf to the dataset

The TEWE distribution is highly flexible, capable of modelling increasing, decreasing, and bathtub-shaped hazard rate function. Simulation findings confirmed the consistency and efficiency of the maximum likelihood estimator. Application to real dataset was demonstrated in Table 2. Fig. 5 is the estimated density plot over histogram for the real life dataset. Fig. 6 is the plot of cdf of the fitted models with empirical cdf. These plots demonstrated the performance of the proposed TE-W-E distribution on the real life datasets. As we can see from Fig. 5 and 6, the TE-W-E model fitted well to the dataset than other fitted competing models and it is confirmed from the results of goodness-of-fit in Table 2. Therefore, TE-W-E model outperformed W, W-E, W-G, Kum-W, TWL, GT-W, TE-W, TNMW, TL and NE-E models on fitting to the dataset which are positively skewed.

CONCLUSION

The TE-W-E distribution is a robust model for lifetime data its flexibility and strong empirical performance make it a valuable contribution to the survival and reliability analysis

FUTURE WORK

The future study focuses on the estimation of the parameters of the proposed TE-W-E distribution using Bayesian estimation (BE) method, compare the MLE with BE methods by checking their Unbiasedness, Efficiency, Mean Square Error, credible and confidence interval for mean of the proposed model as assessment criteria.