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Closed-form expressions and the asymptotics to moments for the excess Gompertz- Makeham distribution

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INTRODUCTION

The distribution was introduced by Gompertz in 1825 with the hazard function

$$h(x)=a\cdot e^{b\cdot x}.$$

It has been proposed in demography to describe the structure of human life expectancy, where a constant b is interpreted as the rate of aging, and a constant *a* as the initial mortality rate. Makeham in 1860 presented the probability distribution as an extension of the Gompertz distribution with the hazard function

$$h\left(x\right)=c+a\cdot e^{b\cdot x}.$$

Here, the constant c is a background constant unrelated to aging, due to age-independent causes. This distribution is called the Gompertz - Makeham distribution now. Reviews on the use of theoretical models in demography are given in the works of Gavrilov and Gavrilova (2011).The distribution of Gomperz - Makeham has a good agreement with the data sets of device failures (Adewara, Adeyeye, Thron (2019)) and the time of detection of software faults (Ohishi, Okamura and Dohi (2009)). A simple closed form for expressing the remaining lifetime was obtained by Castellares, Patrício, Lemonte (2020).

RESULTS & DISCUSSION

Let $F_t(x)$be the conditional distribution function of the remaining lifetimeX_t of a non-negative random variable X with a distribution function $F\ (x)$. The Gompertz-Makeham distribution function has the form

$$F(x;a,b,c)=1-e^{-\left[c\cdot x+\frac{a}{b}\left(e^{b\cdot x}-1\right)\right]}.$$

Theorem. $F_t(x;a,b,c)=F\ (x;\ a\cdot e^{bt},b,c).$

Thus, Gompertz-Makeham distribution preserved for the distribution of remaining lifetime, but with a different first parameter.

According to Theorem , Formulas are obtained by using the G- function for the first, second residual moments and the variance. For example, we have

$$E\left[X_t^2\right]=\frac{2}{b^2}e^{\frac{a\cdot e^{b\cdot t}}{b}}G_{2\,3}^{3\,0}\left(\frac{a\cdot e^{b\cdot t}}{b}\middle|\begin{matrix};\ \frac{c}{b}+1 & \frac{c}{b}+1\\ 0 & \frac{c}{b} & \frac{c}{b}\end{matrix}\right).$$

The asymptotic expansions of the first, second moments and the variance are obtained. So

$$\sigma^2\left(t\right)\simeq\frac{e^{-2bt}}{a^2}-2\left(c+2b\right)\frac{e^{-3bt}}{a^3}+$$

$$\left[2\left(c+3b\right)\left(2c+3b\right)-\left(c+b\right)^2\right]\frac{e^{-4bt}}{a^4}.$$

An error estimate of this formula is given.
The conditions for the existence of parameter estimates using the maximum likelihood estimation for the Gompertz distribution are obtained.

CONCLUSION

There is an oscillation of the variance at large age calculated by the G-function Figure 1:

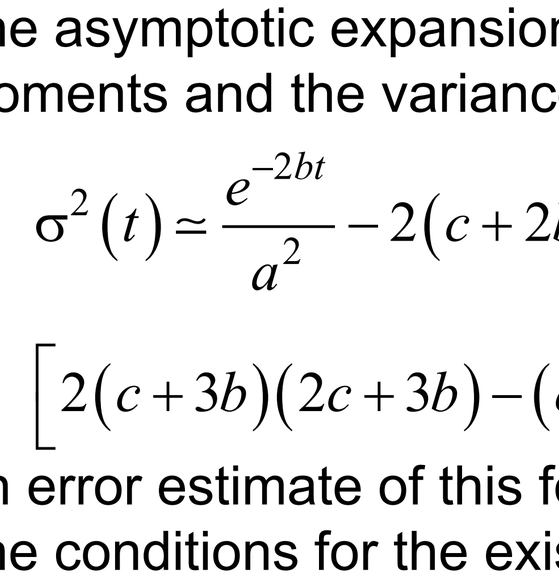

Figure 1. **1**. Variance obtained by G-function; **2**. Variance obtained by asymptotic expansion.

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