The 2nd International Online Conference on Mathematics and Applications

mathematics an Open Access Journal by MDPI

A Nonparametric Approach to Performability Analysis in Semi-Markov Systems

Tayeb HAMLAT [1]

[1] Laboratory of Stochastic Models, Statistic and Applications, Dr. Tahar Moulay University of Saida, Algeria.

Abstract

This work introduces a nonparametric estimator for evaluating the performance of semi-Markov systems, framed as the sum or integral of a real-valued functional stochastic process. For a homogeneous continuous-time semi-Markov process, we derive empirical estimators for the semi-Markov kernel, renewal matrix, semi-Markov transition matrix, and mean performance of the system. We establish asymptotic properties of these estimators, including strong consistency and asymptotic normality. Additionally, the theoretical findings regarding performability are validated through a numerical example.

Introduction

Performability has initially been introduced by Meyer [?] in order to generalize classical reliability indicators. This idea is theoretically stated as the sum or the real-valued integral functional of the process Z for a system with the state space set E, a reward rate function h and a stochastic process Z_t, $t \geq 0$, defined as follows,

$$\Phi(t) = \int_0^t h(Z_u)du, \quad t \geq 0. \tag{1}$$

The performability is the marginal distribution function of the integral functional $\Phi(t)$, $t \geq 0$. It is worth mentioning that (1) has been mainly used in insurance, technological systems, in economical etc. For the purpose of performing semi-Markov systems with general state space, a complete study is given in [5] when Z is homogeneous semi-Markov and continuous-time stochastic process.

Besic set-up

A Markov renewal process is a bivariate stochastic process (J_n, T_n). Let us denote by T_n the nth jump time of the process and J_n the successive visited states of the process. This process has to satisfy the following formula:

$$\begin{aligned}&\mathbb{P}\left(J_{n+1} = j, T_{n+1} - T_n \leq t \mid J_0, J_1, \ldots, J_n, T_0, T_1, \ldots, T_n\right)\\ &= P\left(J_{n+1} = j, T_{n+1} - T_n \leq t \mid J_n\right),\end{aligned}$$

for all $j \in E$, all $t \in \mathbb{R}_+$ and all $n \in \mathbb{N}$.

Continuous-time semi-Markov process

The Semi-Markov Process (SMP) $\{Z_t; t \in \mathbb{R}_+\}$ defined by $Z_t = J_{N(t)}$, where $N(t) = \max\{n \in \mathbb{N} : T_n \leq t\}$ is the counting process of the SMP up to time t. Let us also define $X_{n+1} = T_{n+1} - T_n$ as the successive sojourn times in J_n for any $n \in \mathbb{N}$.

For all $i, j \in E$ let us define:

- The semi-Markov kernel $Q(t) = \left\{Q_{ij}(t), i, j \in E\right\}, t \geq 0$ is given by
$$Q_{ij}(t) = P\left(J_{n+1} = j, X_{n+1} \leq t \mid J_n = i\right).$$
- H_i, the sojourn time distribution in state i.
$$H_i(t) = P\left(X_{n+1} \leq t \mid J_n = i\right) = \sum_{j=1}^{s} Q_{ij}(t), \quad t \in \mathbb{R}_+.$$
- Let us define the Markov renewal function $\Psi_{ij}(t), i, j \in E, t \geq 0$, by
$$\Psi_{ij}(t) = \sum_{n=0}^{\infty} Q_{ij}^{(n)}(t).$$
- We define the semi-Markov transition function $P_{ij}(t), i, j \in E, t \geq 0$, by
$$P_{ij}(t) = \mathbb{P}\left(Z_t = j \mid Z_0 = i\right) = \mathbb{P}\left(J_{N(t)} = j \mid J_0 = i\right).$$
- It is known in [6] that
$$P_{ij}(t) = \mathbf{1}_{\{i=j\}}\left(1 - \sum_{k=1}^{s} Q_{ik}(t)\right) + \sum_{k \in E} \int_0^t P_{kj}(t-s) Q_{ik}(ds).$$

By solving the above Markov renewal equation, cf. [10], the unique solution is given in matrix notation by

$$P(t) = (I - \mathbf{Q}(t))^{(-1)} * (I - \mathrm{diag}(\mathbf{Q}(t)\mathbf{e})) \tag{2}$$

where $\mathrm{diag}(\cdot)$ is a diagonal matrix of i-th entry $\sum_{j=1}^{s} Q_{ij}(t)$ and $e = (1, 1, \ldots, 1)^t$.

Let $(Z_t)_{t \in \mathbb{R}^+}$, be a homogeneous SMP with finite state space E, and let h be a real-valued function defined on E. The performance process is defined by

$$\Phi(t) = \int_0^t h\left(Z_u\right) du = \sum_{i \in E} h(i) \int_0^t \mathbf{1}_{\{z_u = i\}} du, \quad t \geq 0. \tag{3}$$

The mean performance at time $t > 0$, denoted by $\overline{\Phi}(t) := \mathbb{E}[\Phi(t)]$, is given by

$$\overline{\Phi}(t) := \mathbb{E}[\Phi(t)] = \sum_{i \in E} \int_0^t h(i) \mathbb{P}\left[Z_u = i\right] du = \sum_{i \in E} h(i) \int_0^t P_u(i) du. \tag{4}$$

Empirical estimators

Nonparametric estimators of the main characteristics of the SMP Z are defined on sample functions of the MRP over $[0, M]$. These sample functions of the MRP are equivalent to the sample functions $(J_0, J_1, \ldots, J_{N(M)}, X_0, X_1, \ldots, X_{N(M)})$. For all $i, j \in E$, $t \geq 0$ and $t \leq M$, we define:

- The empirical estimator of the semi-Markov kernel $Q_{ij}(t)$ is given by
$$\widehat{Q}_{ij}(t, M) = \frac{1}{N_i(M)} \sum_{n=1}^{N(M)} \mathbf{1}_{\{J_{n-1}=i, J_n=j, X_n \leq t\}}, \tag{5}$$
- The following estimator of the Markov renewal function $\Psi_{ij}(t)$ is given by
$$\widehat{\Psi}_{ij}(t, M) = \sum_{n=0}^{\infty} \widehat{Q}_{ij}^{(n)}(t, M). \tag{6}$$
- The empirical estimator of the transition function of the SMP $P_{ij}(t)$, $i, j \in E$ and $t \geq 0$ is given by the following matrix form
$$\widehat{P}(t, M) = \widehat{\Psi} * (I - \mathrm{diag}(\widehat{Q}(t, M)e)). \tag{7}$$
- Define now the following estimator for the mean performance $\overline{\Phi}_i(t) := \mathbb{E}_i[\Phi(t)]$,
$$\widehat{\overline{\Phi}}_i(t, M) := \sum_{j \in E} h(j) \int_0^t \widehat{P}_{ij}(s, M) ds \tag{8}$$

Asymptotic behavior

Assumptions All along this paper we are working under the following assumptions :

(H.1) The embedded Markov chain $(J_n)_{n \in \mathbb{N}}$ is an ergodic irreducible Markov chain, with stationary distribution ν.

(H.2) The SMP is irreducible, aperiodic, with finite mean sojourn times.

The intermediate following result is needed.

We have the following strong convergence :

$$\frac{\Phi(t)}{t} \xrightarrow{a.s.} \pi h := \textstyle\sum_{i \in E} \pi(i) h(i), \quad t \to +\infty.$$

$$\frac{\overline{\Phi}(t)}{t} \xrightarrow{a.s.} \pi h, \quad t \to +\infty.$$

Using the above definitions and results, the following theorem holds :

The estimator $\widehat{\overline{\Phi}}(t, M)$ of $\overline{\Phi}(t)$ is

- Strongly uniformly consistent, that is
$$\max_{i \in E} \sup_{0 \leq t \leq M} \left|\widehat{\overline{\Phi}}(t, M) - \overline{\Phi}(t)\right| \xrightarrow{a.s.} 0, \quad M \to \infty.$$
- Converges in distribution, for any fixed t, as $M \to \infty$, to a normal random variable, i.e.,
$$M^{1/2}(\widehat{\overline{\Phi}}(t, M) - \overline{\Phi}(t)) \xrightarrow{\mathrm{D}} N\left(0, \sigma_{ij}^2(t)\right),$$

with

$$\begin{aligned}\sigma_{ij}^2(t) = \sum_{i=1}^{s} \sum_{j=1}^{s} \mu_{ii} \Big\{ &\left(W_{ij}\right)^2 * Q_{ij} - \left(W_{ij} * Q_{ij}\right)^2\\ &+ \int_0^\infty \left[\int_0^\infty h(j)(x \wedge (t-u)) dA_i(u)\right]^2 dQ_{ij}(x)\\ &- \left[\int_0^\infty \int_0^\infty h(j)(x \wedge (t-u)) dA_i(u) dQ_{ij}(x)\right]^2\\ &+ 2 \int_0^\infty W_{ij}(t-x) \int_0^\infty h(j)(x \wedge (t-u)) dA_i(u) dQ_{ij}(x)\\ &-2\left(W_{ij} * Q_{ij}\right)(t) \cdot \left(A_i * (h(j)(x \wedge .))\right)(t) \Big\},\end{aligned}$$

where for $t \in \mathbb{R}^+$:

$$A_i(t) = \sum_{k=1}^{s} \alpha_k h(i) \Psi_{ki}(t), \quad \text{and} \quad W_{kl}(t) = \sum_{i=1}^{s} \sum_{j \in U} \alpha_i \left(\Psi_{ik} * \Psi_{lj} * I_j\right)(t).$$

1 Numerical Example

Let us consider a three state semi-Markov system as illustrated in figure 1. States 1 and 2 are up states and state 3 is a down state. The reward rate function h is defined by $h(1) = 1$, $h(2) = 0.6$ and $h(3) = 0$. We have two exponential and two Weibull distribution functions as conditional transitions, for all $x \geq 0$, say $H_{12}(x) = 1 - \exp(-\lambda_1 x)$, $H_{31}(x) = 1 - \exp(-\lambda_2 x)$, $H_{23}(x) = 1 - \exp\left[-\left(\frac{x}{\alpha_1}\right)^{\beta_2}\right]$, $H_{21}(x) = 1 - \exp\left[-\left(\frac{x}{a_1}\right)^{\beta_1}\right]$. The parameters of these distributions are : $\lambda_1 = 0.1$, $\lambda_2 = 0.2$, $\alpha_1 = 0.3$, $\beta_1 = 2$, $\alpha_2 = 0.1$, $\beta_2 = 2$. The transition probability matrix of the embedded Markov chain (J_n) is :

$$P = \begin{pmatrix} 0 & p & 0 \\ p & 0 & 1-p \\ 1 & 0 & 0 \end{pmatrix}$$

where p is given by $p = \int_0^\infty \left[1 - H_{23}(x)\right] dH_{21}(x)$.

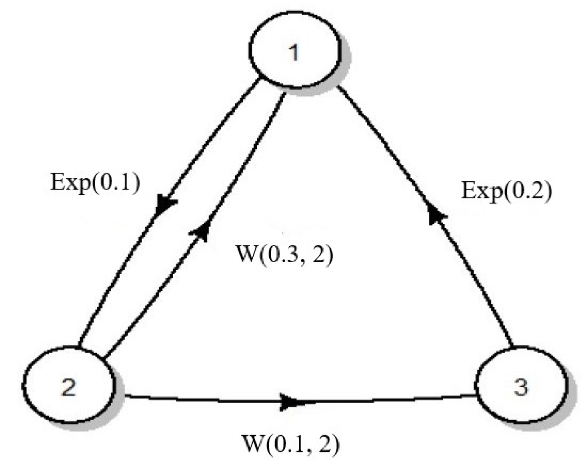

Figure 1: A Three State Semi-Markov System

If we simulate one trajectory for different censure times $M_1 < M_2 < \ldots < M_r$, we see that the performance curve of the estimator (8), that is $\widehat{\overline{\Phi}}_i(t, M_k)$, converges to the true curve of $\overline{\Phi}(t)$ as k increases. Figure 2 illustrates such a situation for two values of M, that is for $M_1 = 100$ and for $M_2 = 1000$. It can be noticed that the second curve is closer to the true curve represented on the same figure 2 by a continuous line.

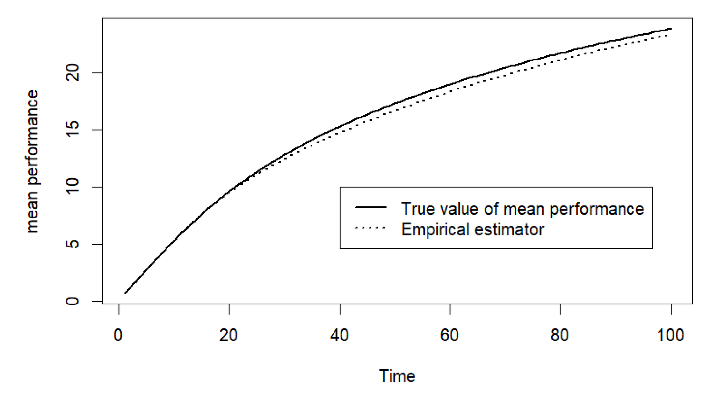

Figure 2: Mean performance estimation of the three state Semi-Markov System

References

[1] Asanjarani, A., Liquet, B. and Nazarathy, Y. (2021). Estimation of Semi-Markov Multi-state models: a Comparison of the Sojourn Times and Transition Intensities Approaches. *The International Journal of Biostatistics.*

[2] Barbu, V. S., Bulla, J. and A. Maroutti. (2012). Estimation of the Stationary Distribution of a Semi-Markov Chain.

[3] Grabski, F. (2015). Semi-Markov Maintenance Nets. Semi-Markov Processes: Applications in System Reliability and Maintenance, pp 217-228.

[4] Janssen, J. and Limnios, N. (1999). Semi-Markov Queues with Heavy Tails, *Semi-Markov Models and Applications. Springer US.* pp 269-284.

[5] Limnios N. and G. Oprişan. (1999). A General Framework for Reliability and Performability Analysis of Semi-Markov Systems, *Appl. Stochastic Models Business Indust.*, vol. 15, no. 4, pp. 353–368.

[6] PYKE, R. (1961). Markov renewal processes: definitions and preliminary properties. *The Annals of Mathematical Statistics*, 1231-1242.

[7] PYKE, R. and R. SCHAUFELE. (1961). Limit theorems for Markov renewal process, *Ann. Math. Statist.*, 35:1746-1764.

[8] OUHBI, B. and N. LIMNIOS. (1996). Non-parametric Estimation for Semi-Markov Kernels with Application to Reliability Analysis. *Applied Stoch. Models Data Anal.*, 12: 209–220.

[9] MOORE, E. H., and PYKE, R. (1968). Estimation of the transition distributions of a Markov renewal process. *Annals of the Institute of Statistical Mathematics*, 20, 411-424.

[10] LIMNIOS, N. and G. OPRIŞAN. (2001). Semi-Markov Processes and Reliability, *Birkhäuser, Boston.*

tayebhamlat1998@gmail.com