The Hamiltonian Degree: A Probabilistic Invariant Connecting Subgroup Normalization and Graph Structure

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Key Insight

Quantitative commutativity measures naturally generate graph-theoretic structures that reveal new algebraic information about finite groups.

Abstract

We define the **Hamiltonian degree**, a new probability measuring how group elements normalize subgroups. Using this measure, we construct a bipartite graph that captures these interactions and compute its degrees and edge structure. Applications to several finite groups illustrate the behavior of this graph. The framework extends classical commutingâĂŠprobability methods and provides a new graphical tool for studying finite groups.

Motivation

- Abelian groups are fully commutative.
- Most finite groups are not.
- Commutativity degrees measure how close a group is to being abelian.
- Graphs provide visual and combinatorial interpretations of algebraic interactions.

Research Question: Can subgroup commutativity generate meaningful graph-theoretic structures?

Historical context

- (Rusin, 1979): Commuting probability

$$d(G) = \frac{|\{(x, y) \in G^2 : xy = yx\}|}{|G|^2}$$

- (Erfanian et al. 2007): Subgroup commutativity degree

$$sd(G) = \frac{|\{(H, K) \in L(G)^2 : HK = KH\}|}{|L(G)|^2}$$

- (Tarnauceanu, 2009): Relative commutativity degree

$$d(H, G) = \frac{|\{(h, g) \in H \times G : hg = gh\}|}{|H||G|}$$

Methodology

Hamiltonian Degree
↓
Normalization Relations
↓
Bipartite Graph Construction
↓
Graph Parameters

Hamiltonian Degree: Definition and Interpretation

Let G be a finite group and $L(G)$ its set of all subgroups. The **Hamiltonian Degree** of G is defined as the probability that a randomly chosen element of G normalizes a randomly chosen subgroup of G:

$$P(G) = \frac{|\{(x, H) \in G \times L(G) : xH = Hx\}|}{|G||L(G)|}.$$

Interpretation: The Hamiltonian Degree measures the likelihood that a random element-subgroup pair satisfies the normalization condition.

Example: Let S_3 be the symmetric group on three letters. Then

$$P(S_3) = \frac{2}{3}.$$

Hamiltonian Degree and Normalizers

Theorem 1

Let $(G, \cdot)$ be a finite group, $L(G) = \{H_1, H_2, \ldots, H_k\}$ the set of all subgroups of G, and $N_G(H_i)$ the normalizer of H_i in G for $i = 1, 2, \ldots, k$. Then

$$P(G) = \frac{\sum_{i=1}^{k} |N_G(H_i)|}{|G||L(G)|}.$$

Hamiltonian Degree and Conjugacy Classes

Theorem 2

Let G be a finite group and $L = \{H_1^G, H_2^G, \ldots, H_k^G\}$ be the set of conjugacy classes subgroups of G. Here, $H_i^G = \{H_i^g : g \in G\}$ is the set of all conjugate subgroups of H_i, $i = 1, 2, \ldots, k$. Then

$$P(G) = \frac{1}{|G||L(G)|} \sum_{j=1}^{k} |H_i^G||N_G(H_i)|.$$

Corollary Let G be a finite group and k the number of distinct conjugacy classes subgroups of G. Then $P(G) = \frac{k}{|L(G)|}$.

Example: The below table represents the Hamiltonian degrees for S_n for $3 \leq n \leq 9$.

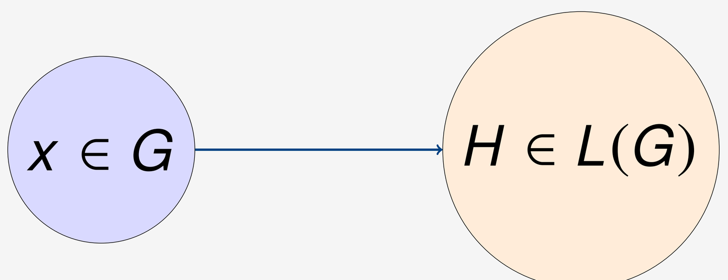
n	$\|S_n\|$	$\|L(S_n)\|$	k	$P(S_n)$
3	6	6	4	$\frac{2}{3}$
4	24	30	11	$\frac{11}{30}$
5	120	156	19	$\frac{19}{156}$
6	720	1455	56	$\frac{56}{1455}$
7	5040	11300	96	$\frac{24}{2825}$
8	40320	151221	296	$\frac{296}{151221}$
9	362880	164723	554	$\frac{554}{164723}$

Key Idea: Hamiltonian-degree methods connect subgroup normalization behavior with global graph-theoretic properties.

Graph Construction

We associate a bipartite graph $\Gamma_P(G)$ to a finite group G as follows:

- The vertex set is $V(\Gamma_P(G)) = G \cup L(G)$, where G consists of group elements and $L(G)$ is the subgroups of G.
- An element $x \in G$ is adjacent to a subgroup $H \in L(G)$ if and only if $xH = Hx$ (equivalently, $x \in N_G(H)$).
- This adjacency encodes subgroup normalization as a combinatorial relation.
- The resulting structure reflects probabilistic interactions between elements and subgroups.



$$xH = Hx \quad \Leftrightarrow \quad x \in N_G(H)$$

Hamiltonian degree transforms subgroup normalization into a bipartite graph whose degrees are completely determined by normalizers.

Main Results

Theorem 3

Let G be a finite group. Then the following statements hold.

(i) For $x \in G$, the degree of x is given by $\deg(x) = P(x, G)|L(G)|$, where $P(x, G) = \frac{|\{H \in L(G) | xH = Hx\}|}{|L(G)|}$.

(ii) For $H \in L(G)$, the degree of H is given by $\deg(H) = |N_G(H)|$.

Interpretation: This theorem converts subgroup-normalization information into computable probabilistic data. Moreover, if G is a Dedekind or Hamiltonian finite group, then $\Gamma_P(G)$ is the complete bipartite graph $K_{|G|,|L(G)|}$.

Theorem 4

The total number of edges in $\Gamma(G)$ is equal to: $|G||L(G)|P(G)$.

Illustration: Let p be an odd prime and D_{2p^n} the Dihedral group with $2p^n$ elements. Then $P(D_{2p^n}) = \frac{2n+2}{n+2+p+p^2+\ldots+p^n}$ and the total number of edges in $\Gamma(D_{2p^n})$ is equal to $(2n+2)2p^n$.

Example: S_3

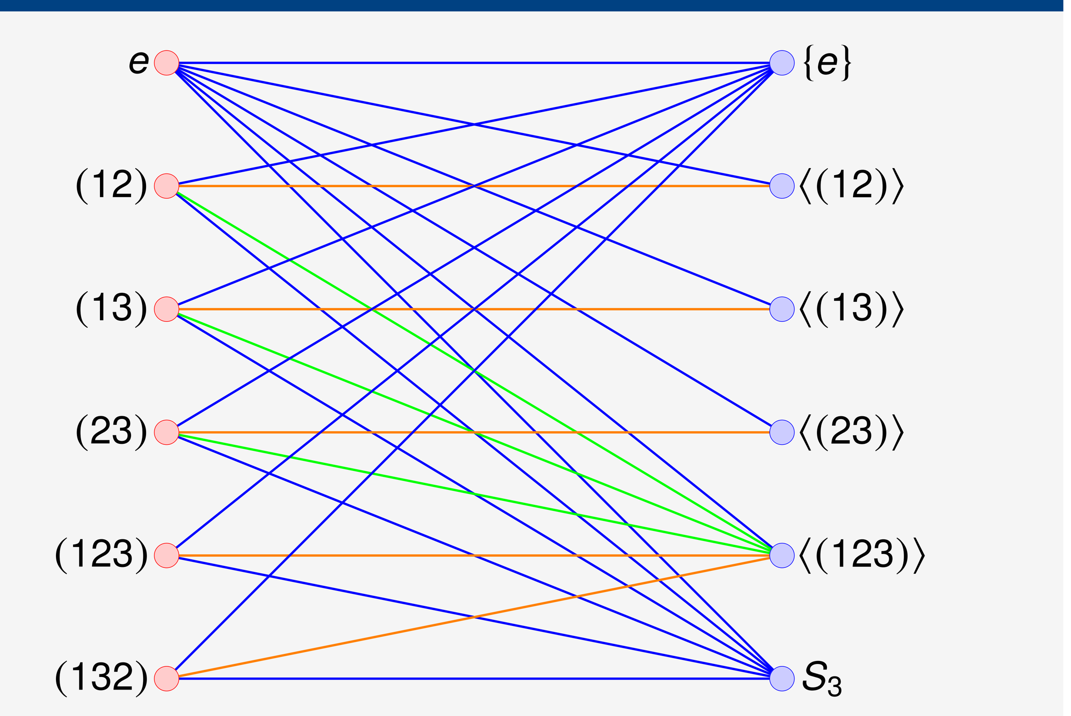


Figure: Graph of $\Gamma_P(S_3)$

Structural Properties	Values
Graph structure	Connected, Not planar, Not complete
Graph invariants	Diameter = 3, Girth = 4, Independence number = 6, Clique number = 2

Comparison with Classical Graphs

Feature	Classical commuting graph	Our graph
Identity vertex	Not included	Included
Connected	Not always	Always
Diameter	Can be large	At most 3
Focus	Element commuting	Subgroup normalizations

Scientific Impact, Conclusion & Future Directions

Scientific Impact

- Bridges probabilistic group theory and graph theory through subgroup normalization.
- Introduces Hamiltonian degree as a computable invariant for finite groups.
- Provides a new graph-theoretic framework for subgroup interactions.

Conclusion

- Hamiltonian degree converts subgroup-element normalization into graph structure.
- The resulting bipartite graph encodes algebraic information in combinatorial form.

Future Directions

- Spectral analysis of the associated graphs.
- Extensions to infinite groups and lattice structures.
- Random walks and probabilistic interpretations.
- Connections with subgroup lattice theory and graph invariants.

Key Message: This framework unifies subgroup normalization and graph structure through a probabilistic invariant.

References

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💡 Quantitative commutativity can actively shape global graph structure.

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