Quadratic Extensions of Hyperfields and Local-Global Principles

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Introduction & Aim

Motivation: Hyperfields arise naturally in quadratic form theory, spaces of orderings, and tropical geometry. They provide an axiomatic setting for the reduced theory of quadratic forms.

Main Goal: Define *quadratic extensions* $H(\sqrt{d})$ for a special hyperfield H such that if $H = Q(F)$ (quadratic hyperfield of a field F), then $H(\sqrt{d})$ is related with $Q(F(\sqrt{d}))$.

Key Idea: Use the Marshall quotient $H/_m(1+d)$ to define the extension. This mirrors the field case where $F(\sqrt{d})$ captures the quadratic extension.

Method: Marshall Quotients & Quadratic Extensions

Definition (Marshall Quotient): For a hyperfield H and a multiplicative set $S \subseteq H$, define an equivalence relation:

$$a \sim b \iff \exists s, t \in S: \ s \cdot a = t \cdot b.$$

Then $H/_m S$ is the quotient hyperfield.

Quadratic Extension: For $d \in H \setminus \{-1\}$, define

$$H(\sqrt{d}) := H/_m(1+d),$$

where $1 + d \subseteq H$. The canonical map is $r_d : H \to H(\sqrt{d})$.

Special Hyperfield Axioms (simplified): A hyperfield is *special* if for all $a \in H^\times$:

$$a^2 = 1, \quad 1 \in 1 + a, \quad \text{and } 1 + a \text{ is closed under multiplication.}$$

Example: $Q(\mathbb{R}) = \{0, 1, -1\}$ with addition:

$$1 + 1 = \{1\}, \quad 1 + (-1) = \{0, 1, -1\}, \quad -1 + (-1) = \{-1\}$$

Result 1: Analogs of Scharlau's Theorems

For a special hyperfield H and $d \in H^\times$ with $d \neq -1$, define:

- $r^* : WH \to WH(\sqrt{d})$ induced by r_d,
- $s_* : WH(\sqrt{d}) \to WH$ by $s_*(\langle \tilde{a}_1, \dots, \tilde{a}_n \rangle) = \langle a_1, \dots, a_n \rangle \otimes \langle 1, d \rangle$,
- $t : WH \to WH$ by $t(\phi) = \langle 1, -d \rangle \otimes \phi$.

Then we have theorems analogous to classical ones:

1. **(Theorem A)** $\langle 1, -d \rangle \otimes WH \subseteq \mathrm{Ker}(r^*)$.
2. **(Theorem B)** $s_*([\phi] \otimes [\psi]) = [\phi] \otimes s_*([\psi])$ (Frobenius–Scharlau reciprocity).
3. **(Theorem C)** Exactness: $Im(s_*) \subseteq \mathrm{Ker}(t)$ and $Im(t) \subseteq \mathrm{Ker}(r^*)$.

These results show that our definition of $H(\sqrt{d})$ correctly captures the arithmetic of quadratic field extensions.

Result 2: n-th Pfister Approximation Property (n-PAP)

Let $B = \{d_i\}_{i \in I} \subseteq H^\times$. (H, B) satisfies **n-PAP** if for all $a_1, \dots, a_n \in H$:

$$\bigcap_{i \in I} D(\langle\langle a_1, \dots, a_n, d_i \rangle\rangle) = D(\langle\langle a_1, \dots, a_n \rangle\rangle).$$

Main Theorem (Quadratic Local-Global Principle):
Let H be an infinite special hyperfield and $B = \{e_i\}_{i \in I}$ an $\mathbb{F}_2$-basis of $H^\times$ satisfying 0-PAP and 1-PAP. Then the map

$$\rho_B : H \to \prod_{i \in I} H(\sqrt{e_i})$$

is a *complete embedding*. Consequently, for forms ϕ, ψ over H:

$$\phi \equiv_H \psi \iff [\phi]_i \equiv_{H(\sqrt{e_i})} [\psi]_i \ \ \forall i \in I.$$

Applications & Examples

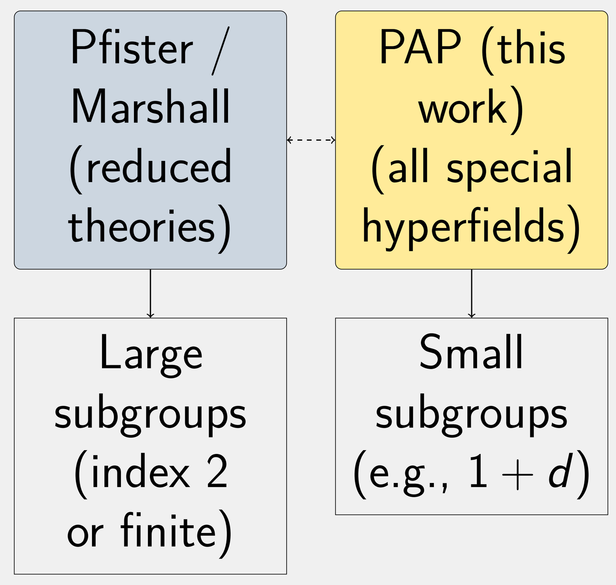
Case 1: Real Reduced Hyperfields (e.g., $Q(\mathbb{R})$)
For any $\mathbb{F}_2$-basis B, (H, B) satisfies 0-PAP and 1-PAP (by separation theorem). Hence ρ_B is a complete embedding.

Case 2: The Rational Numbers $\mathbb{Q}$

- 0-PAP holds for basis $B = \{-1, 2, 3, 5, \dots\}$.
- 1-PAP **fails**: $7 \in \cap_{q \in B} D_{\mathbb{Q}}(\langle\langle 1, q \rangle\rangle)$ but $7 \notin D_{\mathbb{Q}}(\langle 1 \rangle)$.

Field / Hyperfield	PAP properties
Real closed (e.g., $\mathbb{R}$)	n-PAP for all n
Global fields ($\mathbb{Q}$, $\mathbb{F}_q(t)$)	0-PAP only (1-PAP fails)
Reduced special groups	n-PAP for all n (with a suitable basis)

Comparison of Local-Global Principles



Our approach complements classical results by working with *small* subgroups and applies to non-reduced theories.

Conclusion & Future Work

Conclusion:

- Quadratic extensions of hyperfields are well-defined via Marshall quotients.
- They satisfy analogs of Scharlau's key theorems (A, B, C).
- The n-PAP gives a new local-global principle that reflects isotropy for forms over non-reduced special hyperfields.
- The n-PAP is based on small subgroups instead of large (= finite index) subgroups.
- Real reduced hyperfields always satisfy n-PAP (for a suitable basis), while global fields like $\mathbb{Q}$ only satisfy 0-PAP.

Future Directions:

1. Connect to graded Witt rings, K-theory, and cohomology rings.
2. Explore the corestriction property of s_* and connecting morphisms.
3. Study non-reduced hyperfields that still satisfy PAP (e.g., via Pfister subgroups).
4. Apply to the classification of Witt rings of function fields.

Key References

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