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On the Upper Embeddability of Johnson Graphs $J(n,k)$ for $n > k \geq 2$

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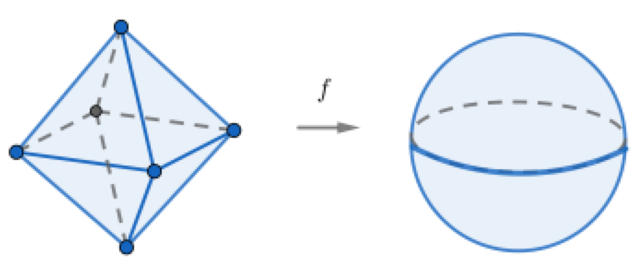
1. INTRODUCTION & AIM

- Topological graph embeddings, particularly 2-cell embeddings, provide an important connection between graph theory and surface topology.



- For a connected graph G, the maximum genus denotes the largest genus of an orientable surface on which G admits a 2-cell embedding.
- A well-known upper bound for the maximum genus is

$$\gamma_M(G) \leq \left\lfloor \frac{\beta(G)}{2} \right\rfloor,$$

where $\beta(G)$ is the *Betti number* of G.
- Graphs attaining this bound are called *upper embeddable* graphs.

1.1 Johnson graph ($J(n,k)$)

- Vertices are the k-element subsets of the base set $\{1,2,3,\ldots,n\}$.
- Two vertices are adjacent if they share exactly $k-1$ elements.

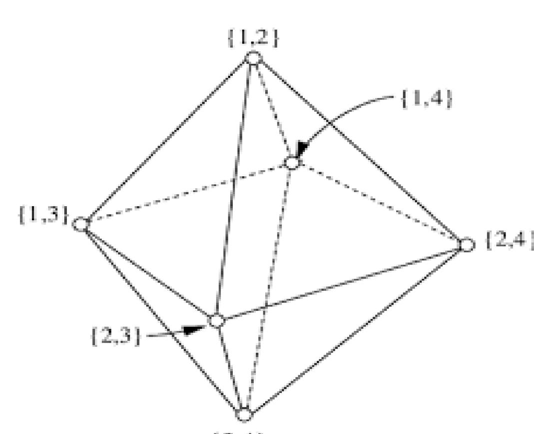


Figure 1. Johnson Graph $J(4,2)$

The main objectives of the study are to:
- Prove the upper embeddability of the Johnson graph $J(n,k)$ for $n > k \geq 2$.
- Derive a formula for its corresponding maximum genus.

2. METHOD

Introduction of lexicographic labeling for the vertices and edges of $J(n,k)$

Construction of a spanning tree

Proof of the splitting tree characterization

Establishment of the upper embeddability of $J(n,k)$ using *Jungerman's* theorem.

Derivation of an explicit formula for the maximum genus.

3. RESULTS & DISCUSSION

3.1 Construction of the Spanning Tree

The spanning tree is constructed using the following steps.
- Label the vertices as $v_1, v_2, v_3, .., v_{{}^nC_k}$ in lexicographic order.
- Arrange the edges in the order
$(v_i, v_j) < (v_r, v_s)$ if $i < r$ or $(i = r$ and $j < s)$.
- Starting with the trivial graph with all vertices, Construct the spanning tree by sequentially adding edges from the above list in the given order while avoiding cycles.

With this construction, the vertex v_1 attains the maximum degree $k(n-k)$ in the spanning tree of $J(n,k)$.

3.2 Upper Embeddability

Lemma 1. Any two distinct vertices are connected by more than two internally disjoint paths in the Johnson graph $J(n,k)$ with $n > k \geq 2$.

Theorem 1. The edge–complement of the spanning tree in section 3.1 of the Johnson graph $J(n,k)$ consists of exactly two components: one isolated vertex and one connected subgraph.

Theorem 2. (Main Result**)** The Johnson graph $J(n,k)$, for $n > k \geq 2$, is upper embeddable. Furthermore, the maximum orientable genus is given by

$$\gamma_M(G) = \left\lfloor \frac{k(n-k)-2}{2k!} \prod_{i=0}^{k-1}(n-i) \right\rfloor.$$

Example: $J(4,2)$

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$$

$$E = \{e_1, e_2, e_3, e_4, \ldots, e_{12}\} = \{(v_1,v_2),(v_1,v_3),(v_1,v_4),(v_1,v_5),..,(v_5,v_6)\}$$

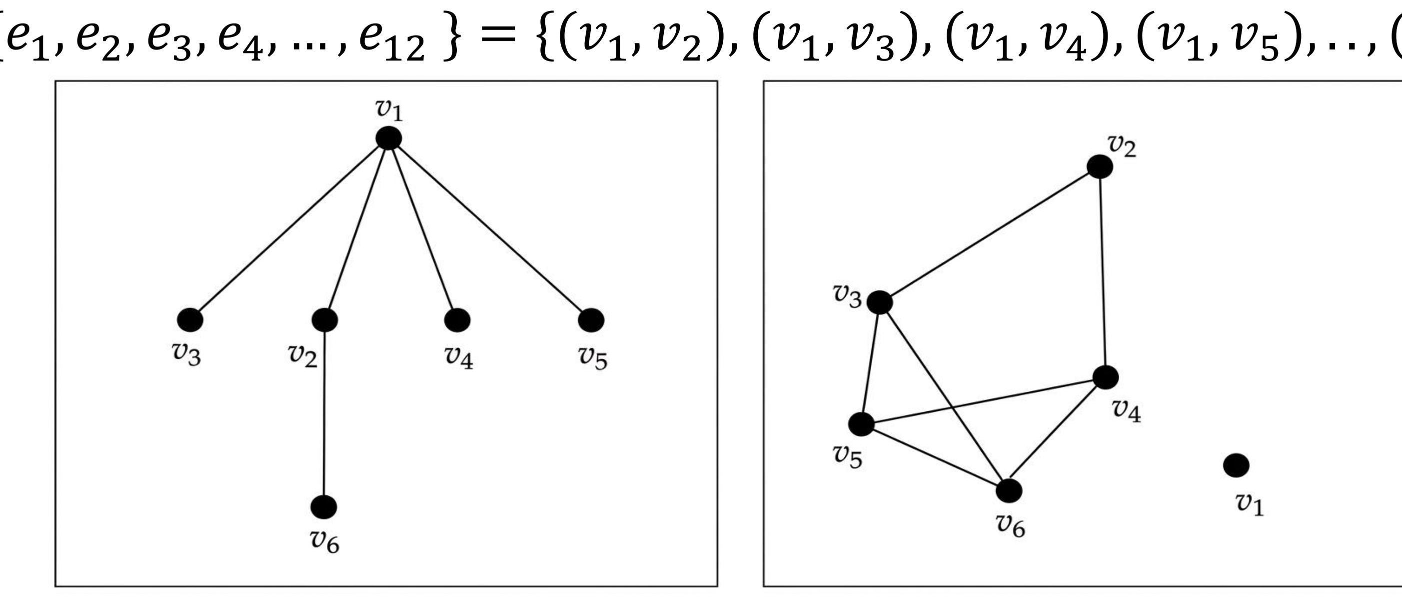


Figure 2. Spanning Tree and Edge complement of $J(4,2)$

By the Theorem 2, the graph $J(4,2)$ is upper embeddable and the maximum genus is

$$\gamma_M(J(4,2)) = \left\lfloor \frac{2(4-2)(4-1)}{2.2!} \right\rfloor = 3.$$

4. CONCLUSION & FUTURE WORK

The Johnson graph $J(n,k)$, for $n > k \geq 2$, is upper embeddable and the maximum orientable genus is given by

$$\gamma_M(G) = \left\lfloor \frac{k(n-k)}{2k!} \prod_{i=1}^{k-1}(n-i) \right\rfloor.$$

The next step is to find the minimum genus and establish a range for the orientable genus of the Johnson graph $J(n,k)$.

5. REFERENCES

- M. Jungerman, "A characterization of upper-embeddable graphs," Transactions of the American Mathematical Society, vol. 241, pp. 401–406, 1978.
- C. A. Smith, "Embeddings of harary graphs in orientable surfaces," master's thesis, University of Maine, 2019.