

On the topology induced by fuzzy partial metric spaces

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INTRODUCTION & AIM

In recent years, fuzzy metric spaces (Kramosil and Michálek, 1975; George and Veeramani, 1994) and fuzzy partial metric spaces (Gregori et al., 2019) have attracted considerable attention due to their ability to generalize classical metric structures and model uncertainty in a more flexible and nuanced manner. In particular, fuzzy partial metrics extend fuzzy metrics by allowing different self-distances for points. Considering that each point's self-distance may vary in fuzzy partial metric spaces, the concept of an open ball was defined accordingly, differing from its definition in classical fuzzy metric spaces. One of the central problems in this area is the construction and comparison of topologies induced by different notions of open balls. Gregori et al. (2019) defined open balls in two different ways and then showed that a topology can be induced using only one of these concepts. Then, Wang et al. (2025) studied some properties of this induced topology, such as separation axioms, boundedness and compactness.

In this study, we introduce a new notion of open balls in fuzzy partial metric spaces and demonstrate that a topology can be constructed from a fuzzy partial metric via the proposed concept. We further compare this notion with other variants of open balls and examine the corresponding induced topologies. Additionally, we investigate fundamental properties of this topology, including metrizable and compactness, providing insight into how classical topological properties extend to the fuzzy partial metric setting.

METHOD

Definition 1 (Gregori et al., 2019): Let $X \neq \emptyset$ be a set, $*$ be a continuous t-norm and $P: X \times X \times (0, \infty) \rightarrow [0,1]$ be a mapping. If the following conditions are satisfied for all $x, y, z \in X$ and $t, s > 0$, then the triple $(X, P, *)$ is called fuzzy partial metric space:

- (FPM1) $0 < P(x, y, t) \leq P(x, x, t)$,
- (FPM2) $P(x, x, t) = P(y, y, t) = P(x, y, t) \Leftrightarrow x = y$,
- (FPM3) $P(x, y, t) = P(y, x, t)$,
- (FPM4) $(P(x, x, t) \rightarrow_* P(x, y, t)) * (P(y, y, s) \rightarrow_* P(y, z, s)) \leq P(x, x, t + s) \rightarrow_* P(x, z, t + s)$.
- (FPM5) $P(x, y, \cdot): (0, \infty) \rightarrow [0,1]$ is continuous.

Here, " \rightarrow_* " denotes the residuum operator related to the t-norm $*$.

Example 2 (Gregori et al., 2019): Let (X, p) be a partial metric space, $a *_p b = ab$ and

$$a *_H b = \begin{cases} 0, & a = b = 0 \\ \frac{ab}{a+b-ab}, & \text{otherwise} \end{cases} \quad \text{for all } a, b \in [0,1]. \quad \text{Define the mappings}$$

$P_e: X \times X \times (0, \infty) \rightarrow [0,1]$ and $P_d: X \times X \times (0, \infty) \rightarrow [0,1]$ as $P_e(x, y, t) = e^{-\frac{p(x,y)}{t}}$ and $P_d(x, y, t) = \frac{t}{t+p(x,y)}$ for all $x, y \in X$ and $t > 0$. Then, $(X, P_e, *_p)$ and $(X, P_d, *_H)$ are fuzzy partial metric spaces.

Definition 3 (Aygün et al., 2022): Let $(X, P, *)$ be a fuzzy partial metric space, (x_n) be a sequence in X and $x \in X$.

- (a) (x_n) is said to be convergent to $x \in X$ if $\lim_{n \rightarrow \infty} P(x_n, x, t) = P(x, x, t)$ for all $t > 0$.
- (b) (x_n) is said to be a Cauchy if $\lim_{n, m \rightarrow \infty} P(x_n, x_m, t)$ exists and greater than zero for all $t > 0$.
- (c) $(X, P, *)$ is called complete if each Cauchy sequence in X converges to a point $x \in X$ and $\lim_{n, m \rightarrow \infty} P(x_n, x_m, t) = \lim_{n \rightarrow \infty} P(x_n, x, t) = P(x, x, t)$ for all $t > 0$.

Definition 4 (Gregori et al., 2019): Let $(X, P, *)$ be a fuzzy partial metric space, $x \in X$, $r \in (0,1)$ and $t > 0$. Then, the open balls centered at x with radius r and parameter t are defined as follows:

$$\begin{aligned} \widetilde{B}_P(x, r, t) &= \{y \in X \mid \widetilde{P}_{x,x,y}(t) > 1 - r\}, \\ B_P(x, r, t) &= \{y \in X \mid P_{x,x,y}(t) > 1 - r\} \end{aligned}$$

where $\widetilde{P}_{x,x,y}(t) = P(x, x, t) \rightarrow_* P(x, y, t)$ and $P_{x,x,y}(t) = \sup\{P(x, x, s) \rightarrow_* P(x, y, s) \mid s \in (0, t)\}$.

It can be observed from the axiom (FPM4) that the mapping $\widetilde{P}_{x,x,y}: (0, \infty) \rightarrow [0,1]$ is non-decreasing. However, the mapping $\widetilde{P}_{x,x,y}: (0, \infty) \rightarrow [0,1]$ may not be left-continuous as shown in Example 4.2 given by Gregori et al. (2019). In contrast, the mapping $P_{x,x,y}: (0, \infty) \rightarrow [0,1]$ is left-continuous and the following inequality is satisfied:

$$P_{x,x,z}(t+s) \geq P_{x,x,y}(t) * P_{y,y,z}(s).$$

In addition, if the mapping $\widetilde{P}_{x,x,y}: (0, \infty) \rightarrow [0,1]$ is left-continuous, then $\widetilde{B}_P(x, r, t) = B_P(x, r, t)$. Furthermore, the family $\mathcal{B} = \{B_P(x, r, t) \mid x \in X, r \in (0,1), t > 0\}$ is a basis for a topology on X . This topology is denoted by τ_P .

Another approach given by Aygün et al. (2022) to induce a topology involves generating a fuzzy metric from a given fuzzy partial metric and then examining the resulting topology:

Proposition 5 (Aygün et al., 2022): Let $(X, P, *)$ be a fuzzy partial metric space satisfying (FPM4*) $P(x, y, t) * (P(y, y, s) \rightarrow_* P(y, z, s)) \leq P(x, z, t + s)$

for all $x, y, z \in X$ and $t, s > 0$. Define the mapping $M_P: X \times X \times (0, \infty) \rightarrow [0,1]$ as

$$M_P(x, y, t) = \begin{cases} P(x, y, t), & x \neq y \\ 1, & x = y. \end{cases}$$

Then, $(X, M_P, *)$ is a fuzzy metric space and also the topology τ_{M_P} induced by M_P is finer than the topology τ_P induced by P . i.e., $\tau_P \subseteq \tau_{M_P}$.

Recently, Zheng and He (2025) considered the mapping $M'_P: X \times X \times (0, \infty) \rightarrow [0,1]$ defined by $M'_P(x, y, t) = (P(x, x, t) \rightarrow_* P(x, y, t)) \wedge (P(y, y, t) \rightarrow_* P(y, x, t)) = \widetilde{P}_{x,x,y}(t) \wedge \widetilde{P}_{y,y,x}(t)$ and claimed that $(X, M'_P, *)$ is a fuzzy metric space. However, it fails that $M'_P(x, y, \cdot): (0, \infty) \rightarrow [0,1]$ is continuous

RESULTS & DISCUSSION

We propose the above approach for constructing a topology from a given fuzzy partial metric space. To this, we define the open ball as follows:

Definition 6: Let $(X, P, *)$ be a fuzzy partial metric space, $x \in X, r \in (0,1)$ and $t > 0$. Then, the open ball centered at x with radius r and parameter t is defined as follows:

$$B'_P(x, r, t) = \{y \in X \mid \widetilde{P}_{x,x,y}(t) \wedge \widetilde{P}_{y,y,x}(t) > 1 - r\}.$$

It is clear from the definitions that $B_P(x, r, t) \subseteq \widetilde{B}_P(x, r, t)$ and $B'_P(x, r, t) \subseteq \widetilde{B}_P(x, r, t)$. While no direct inclusion relation between $B_P(x, r, t)$ and $B'_P(x, r, t)$ follows immediately from the definitions, we provide the following example satisfying the inclusion $B'_P(x, r, t) \subseteq B_P(x, r, t)$.

Example 7: Let $X = [0, \infty)$, $p(x, y) = \max\{x, y\}$ and $(X, P_d, *_H)$ be the fuzzy partial metric space given in Example 2. Here,

$$P_d(x, x, t) \rightarrow_{*_H} P_d(x, y, t) = \frac{t}{t + \max\{x, y\} - \max\{x, x\}}$$

for all $x, y \in X$ and $t > 0$. Then, we obtain

$$(P_d(x, x, t) \rightarrow_* P_d(x, y, t)) \wedge (P_d(y, y, t) \rightarrow_* P_d(x, y, t)) = \frac{t}{t + |x - y|}$$

and also $B_P(x, r, t) = [0, x + \frac{rt}{1-r}] \subseteq X$ and $B'_P(x, r, t) = (x - \frac{rt}{1-r}, x + \frac{rt}{1-r}) \subseteq X$ for all $x \in X, r \in (0,1)$ and $t > 0$

Proposition 8: Let $(X, P, *)$ be a fuzzy partial metric space, then the family $\mathcal{B}' = \{B'_P(x, r, t) \mid x \in X, r \in (0,1), t > 0\}$

is a basis for a topology on X and we denote this topology by τ'_P .

Again, Example 6 suggests that $\tau_P \subseteq \tau'_P$ and also $\tau'_P = \tau_e$ where τ_e is the topology generated by Euclidean metric.

Proposition 9: The topological space (X, τ'_P) is a T_1 -space.

Proposition 10: Let $(X, P, *)$ be a fuzzy partial metric space and $U_n = \left\{ (x, y) \in X \times X \mid \widetilde{P}_{x,x,y}\left(\frac{1}{n}\right) \wedge \widetilde{P}_{y,y,x}\left(\frac{1}{n}\right) > 1 - \frac{1}{n} \right\}$

for all $n \in \mathbb{N}$. Then, the family $\mathcal{U} = \{U_n \mid n \in \mathbb{N}\}$ is a basis for a uniformity on X whose induced topology coincides with τ'_P .

Proposition 11: The topological space (X, τ'_P) is metrizable.

Proposition 12: Let $(X, P, *)$ be a complete fuzzy partial metric space, then the topological space (X, τ'_P) is completely metrizable.

Proposition 13: (X, τ'_P) is compact if and only if (X, τ_P) is sequentially compact.

Proposition 14: (X, τ'_P) is compact if and only if (X, \mathcal{U}) is complete and totally bounded.

CONCLUSION

In this paper, we introduce a novel open ball in fuzzy partial metric space and show that the family of these open balls induces a topology. Then, we study some properties of this topology. These results demonstrate that the topology (τ'_P) provides a natural and well-behaved topological framework. The construction establishes a bridge between fuzzy partial metric structures with topological and uniform spaces, opening the possibility for further investigations of separation axioms, compactness conditions, completeness properties, and fixed point theory within this setting.

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