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Annihilator-Based Dependency in Module Theory

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INTRODUCTION & AIM

Problem Statement

Classical submodule dependency relies on linear or essential dependence, missing deeper interactions between annihilator ideals and radical structure. This work introduces a radical-algebraic dependency framework.

Key Definitions

Def 1 — Radically Dependent Submodules

$N_1, N_2 \le M$ are radically dependent, written $N_1 \rhd_{rad} N_2$, if:

$$\sqrt{Ann(N_1+N_2)} = \sqrt{Ann(N_1)} + \sqrt{Ann(N_2)}$$

Symmetric relation. Captures prime ideal structure without full annihilator equality.

Def 2 — Totally Annihilator-Dependent Module

M is totally ann.-dependent if for all nonzero $N_1, N_2 \le M$ with $Ann(N_i) \neq 0$:

$$\dim R/Ann(N_1+N_2) = \min\{ \dim R/Ann(N_1), \dim R/Ann(N_2) \}$$

Def 3 — Radical Distinction Sets

$$Z\_R(M) = \{ m \in M\setminus\{0\} \mid Ann(Rm) \neq Ann(Rad(M)) \}$$
$$\tilde{Z}\_R(M) = \{ m \in M\setminus\{0\} \mid \sqrt{Ann(Rm)} \neq \sqrt{Ann(Rad(M))} \}$$

Z_R uses exact equality; Z̃_R uses radical equality — a coarser partition.
When Z_R(M)=∅, every nonzero m satisfies Ann(Rm)=Ann(Rad(M)).

Annihilator Radical Equivalence Relation ~_g

Define on $M\setminus\{0\}$: $m_1 \sim_g m_2$ iff $\sqrt{Ann(Rm_1)} = \sqrt{Ann(Rm_2)}$.
Equivalence class: $[m]_g = \{ m' \in M\setminus\{0\} \mid \sqrt{Ann(Rm')} = \sqrt{Ann(Rm)} \}$.

This classifies elements by their radical annihilator type, giving a finer structural decomposition than Z_R(M) alone.

Goals

① Characterize totally ann.-dependent modules via Ass(M).
② Connect Z_R(M)=∅ to singleton associated prime sets.
③ Prove a biconditional radical sum equivalence (Thm 3).
④ Extend results to f.g. multiplicative modules (Thm 4).

METHOD

Proof Strategies

Thm 1 — Forward ($Ass(M) = \{\mathfrak{p}\} \Rightarrow$ totally ann.-dep.)

Since $Ass(N_i) \subseteq Ass(M) = \{\mathfrak{p}\}$ and $Ass(N_i) \neq \emptyset$, we get $Ass(N_i) = \{\mathfrak{p}\}$, so $\sqrt{Ann(N_i)} = \mathfrak{p}$.
$Ann(N_1+N_2) = Ann(N_1) \cap Ann(N_2)$, so:

$$\sqrt{Ann(N_1+N_2)} = \mathfrak{p}\cap\mathfrak{p} = \mathfrak{p} \Rightarrow \dim R/Ann(N_1+N_2) = \dim R/\mathfrak{p} \ ✓$$

Thm 1 — Reverse (totally ann.-dep. $\Rightarrow Ass(M) = \{\mathfrak{p}\}$)

If $\mathfrak{p}_1 \neq \mathfrak{p}_2$ both in Ass(M), take $N_i = Rm_i$ with $Ann(m_i) = \mathfrak{p}_i$ (ann. of assoc. primes).
Since $\mathfrak{p}_1\cap\mathfrak{p}_2 \subsetneq \mathfrak{p}_1$, the chain $\mathfrak{p}_1\cap\mathfrak{p}_2 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_d$ has length d+1:

$$\dim R/(\mathfrak{p}_1\cap\mathfrak{p}_2) \ge d+1 > \min\{d,d\}$$ — contradicts tot. ann.-dep. ■

Thm 2 — $Z\_R(M) = \emptyset$ Proof

Every $m \neq 0$ satisfies $Ann(Rm) = Ann(Rad(M))$ (by $Z\_R(M)=\emptyset$).
Every associated prime of M equals Ann(Rad(M)), so Ass(M) = {Ann(Rad(M))}, which must be prime.

Thm 3 — Radical Sum Equivalence (both directions)

(⟹) If $Ass(M) \supseteq \{\mathfrak{p}_1,\mathfrak{p}_2\}$ with $\mathfrak{p}_1 \neq \mathfrak{p}_2$, take $N_i = Rm_i$. Then:

$$\sqrt{Ann(N_1+N_2)} \subseteq \mathfrak{p}_1\cap\mathfrak{p}_2 \subsetneq \mathfrak{p}_1+\mathfrak{p}_2 = \sqrt{Ann(N_1)}+\sqrt{Ann(N_2)}$$ — contradiction.

(⟸) $Ass(M)=\{\mathfrak{p}\} \Rightarrow Ass(N_i)=\{\mathfrak{p}\} \Rightarrow \sqrt{Ann(N_i)}=\mathfrak{p} \Rightarrow$ both sides equal $\mathfrak{p}$.

Thm 4 — Multiplicative Modules

M multiplicative means every submodule N = IM for some ideal I. Then $Ann(N) = \{r \mid rI \subseteq Ann(M)\}$.
$Ann(N_1+N_2) = Ann(N_1)\cap Ann(N_2)$ still holds, and $Ass(N_i)=\{\mathfrak{p}\}$ gives:

$$\sqrt{Ann(N_1+N_2)} = \mathfrak{p}\cap\mathfrak{p} = \mathfrak{p} = \sqrt{Ann(N_1)}+\sqrt{Ann(N_2)} \ ✓$$

RESULTS & DISCUSSION

Main Theorems

Theorem 1 — Characterization (R Noetherian, M finitely generated)

M totally ann.-dependent ⟺ $Ass(M) = \{\mathfrak{p}\}$ for a unique prime $\mathfrak{p}$

Key: Krull dimension of $R/Ann(N_i)$ is determined by $\sqrt{Ann(N_i)} = \mathfrak{p}$.

Theorem 2 — $Z\_R(M) = \emptyset$ Implies Singleton Ass

$$Z\_R(M) = \emptyset \Rightarrow Ass(M) = \{ Ann(Rad(M)) \}$$

and Ann(Rad(M)) is a prime ideal.
Corollary 5: Z_R(M) = ∅ ⟹ M is totally ann.-dependent.

Theorem 3 — Radical Sum Equivalence (biconditional)

For all nonzero $N_1, N_2 \le M$:

$$[\ \sqrt{Ann(N_1+N_2)} = \sqrt{Ann(N_1)}+\sqrt{Ann(N_2)}\ ] \iff Ass(M) = \{\mathfrak{p}\}$$

Theorem 4 — Multiplicative Modules (f.g. multiplicative, $Ass(M)=\{\mathfrak{p}\}$)

$$\sqrt{Ann(N_1+N_2)} = \sqrt{Ann(N_1)} + \sqrt{Ann(N_2)} = \mathfrak{p} \text{ for all } N_1,N_2 \le M$$

Uses N = IM structure; Ann(IM) = {r | rI ⊆ Ann(M)}.

Supporting Lemmas

Lemma 3: $N_1 \le N_2 \Rightarrow \sqrt{Ann(N_2)} \subseteq \sqrt{Ann(N_1)}$ (radical reverses inclusion)
Lemma 4: $\sqrt{Ann(N_1)} = \sqrt{Ann(N_2)} \Rightarrow N_1 \rhd_{rad} N_2$ (equal radicals ⟹ radically dependent)
Proof of Lemma 4: $\sqrt{Ann(N_1+N_2)} = \sqrt{Ann(N_1)\cap Ann(N_2)} = \mathfrak{p}\cap\mathfrak{p} = \mathfrak{p} = \mathfrak{p}+\mathfrak{p}$.

Radical Distinction via Examples

Z_R(M) and Z̃_R(M) can differ — the first detects exact annihilator deviation, the second only radical deviation.

Example 8: $M = \mathbb{Z}/8\mathbb{Z}$

$Rad(M) = 2\mathbb{Z}/8\mathbb{Z}$, $Ann(Rad(M)) = 4\mathbb{Z}$
$Ann(\bar{1})=Ann(\bar{3})=Ann(\bar{5})=Ann(\bar{7})=8\mathbb{Z}$, $Ann(\bar{2})=Ann(\bar{6})=4\mathbb{Z}$, $Ann(\bar{4})=2\mathbb{Z}$

$Z\_R(M) = \{\bar{1}, \bar{3}, \bar{4}, \bar{5}, \bar{7}\}$ but $\tilde{Z}\_R(M) = \emptyset$

All elements have $\sqrt{Ann(\bar{m})}=2\mathbb{Z}=\sqrt{Ann(Rad(M))}$, so Z̃_R(M)=∅ even though Z_R(M)≠∅.

Example 9: $M = \mathbb{Z}/5\mathbb{Z}$ (simple module)

$Rad(M) = 0$, $Ann(Rad(M)) = \mathbb{Z}$. For all $m \neq \bar{0}$: $Ann(Rm) = 5\mathbb{Z} \neq \mathbb{Z}$.

$Z\_R(M) = \tilde{Z}\_R(M) = M\setminus\{\bar{0}\} = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$

Both sets coincide for simple modules (semisimple ⟹ Rad(M)=0).

Positive Case: $M = \mathbb{Z}_{25}$ (f.g. multiplicative, Theorem 4 ✓)

$Ass(M) = \{(5)\}$. For every nonzero $N \le M$: $\sqrt{Ann(N)} = 5\mathbb{Z} = (5)$.

$$\sqrt{Ann(N_1+N_2)} = \sqrt{Ann(N_1)} + \sqrt{Ann(N_2)} = 5\mathbb{Z} \ ✓$$

Confirms Theorem 4 concretely: single prime, radical formula holds.

Limiting Case: $M = E(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}_{p^\infty}$ (Prüfer p-group)

Divisible, not f.g., not multiplicative. $Ann(N_1)=p\mathbb{Z}$, $Ann(N_2)=0 \Rightarrow Ann(N_1+N_2)=0$.

$$\sqrt{Ann(N_1+N_2)} = 0 \neq p\mathbb{Z} = \sqrt{Ann(N_1)}+\sqrt{Ann(N_2)}$$

CONCLUSION

① Single associated prime ⟺ radical sum formula — a clean biconditional duality.
② Z_R(M)=∅ forces Ass(M) to be a singleton and Ann(Rad(M)) to be prime.
③ The equivalence relation ~_g provides a finer annihilator-type classification of module elements.
④ Finiteness + multiplicativity are both essential for Theorem 4; the Prüfer group shows both can fail.

FUTURE WORK / REFERENCES

Future Work

Extend to non-Noetherian rings; investigate Z_R(M) in the context of flat modules, injective hulls, and projective dimension.

References

Khaksari, Sharif & Ershad (2004) | Lee, Moo & Varmazyar (2017) | Koc (2025) | Hassanzadeh (2016) | Smith (2004)