

SOME GENERALIZED SEQUENCE SPACE DEFINE BY MUSIELAK-ORLICZ FUNCTION

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Introduction & Objective

The sequence space $M(\phi)$ was introduced by Winifred L.C. Sargent in the 19th century. An Orlicz function $\pi : [0, \infty) \rightarrow [0, \infty)$ is a continuous function that is non-decreasing and convex. A sequence $\Pi = (\pi_k)$ of Orlicz functions is called a Musielak-Orlicz function.

Research Objective: To introduce and investigate new classes of sequence spaces by combining the structural flexibility of Musielak-Orlicz functions with generalized averaging techniques over a linear space (V, q) equipped with a seminorm q .

Methodology

The research employs a deductive approach grounded in functional analysis and sequence space theory:

- **Axiomatic Construction:** We define the new sequence spaces $l_1[\Pi, q]$, $l_\infty[\Pi, q]$, and $M[\Pi, \phi, q]$ as generalizations of the classical $M(\phi)$ space using a Musielak-Orlicz function $\Pi = (\pi_k)$.
- **Topological Analysis:** We construct suitable Luxemburg-type seminorms to establish the topological structures of these spaces.
- **Analytical Proofs:** We apply standard analytical techniques to prove fundamental structural properties including linearity, solidity, monotonicity, and completeness.

Proposed Sequence Spaces

Let $\Pi = (\pi_k)$ be a Musielak-Orlicz function. We introduce the following spaces:

$$l_1[\Pi, q] = \left\{ (x_k) \in \omega : \sum_{k=1}^{\infty} \pi_k \left(\frac{q(x_k)}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}$$

$$l_\infty[\Pi, q] = \left\{ (x_k) \in \omega : \sup_{k \geq 1} \pi_k \left(\frac{q(x_k)}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}$$

$$M[\Pi, \phi, q] = \left\{ (x_k) \in \omega : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} \pi_k \left(\frac{q(x_k)}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}$$

Main Results: Algebraic & Topological Properties

Theorem 1, 2, 3 (Linearity): The spaces $M[\Pi, \phi, q]$, $l_1[\Pi, q]$, and $l_\infty[\Pi, q]$ are linear spaces.

Theorem 4, 5, 6 (Seminorms): $M[\Pi, \phi, q]$ is a seminormed space with the seminorm:

$$h((x_k)) = \inf \left\{ \rho > 0 : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} \pi_k \left(\frac{q(x_k)}{\rho} \right) \leq 1 \right\}$$

Similarly, $l_1[\Pi, q]$ and $l_\infty[\Pi, q]$ are seminormed spaces.

Theorem 7 (Completeness): If the space (V, q) is complete, then $M[\Pi, \phi, q]$, $l_1[\Pi, q]$, and $l_\infty[\Pi, q]$ are complete with their respective seminorms.

Theorem 8 & Corollary 1: The space $M[\Pi, \phi, q]$ is solid, symmetric, and monotone.

Proposition 1: The spaces $l_1[\Pi, q]$ and $l_\infty[\Pi, q]$ are solid and as such are monotone.

Main Results: Inclusion Relations

Theorem 9: $M[\Pi, \phi, q] \subseteq M[\Pi, \psi, q]$ if and only if $\sup_{s \geq 1} \frac{\phi_s}{\psi_s} < \infty$.

Theorem 10: Suppose $\Pi_1 = (\pi_k^1)$ and $\Pi_2 = (\pi_k^2)$ are Musielak-Orlicz functions satisfying the Δ_2 -condition. Then:

- $M[\Pi_1, \phi, q] \subseteq M[\Pi_2 \circ \Pi_1, \phi, q]$
- $M[\Pi_1, \phi, q] \cap M[\Pi_2, \phi, q] \subseteq M[\Pi_1 + \Pi_2, \phi, q]$

Theorem 11: $l_1[\Pi, q] \subseteq M[\Pi, \phi, q] \subseteq l_\infty[\Pi, q]$

Special Cases & Examples

If we restrict the Musielak-Orlicz function $\Pi = (\pi_k)$ such that $\pi_k(x) = \pi(x)$ for all k , and let $q(x) = |x|$, the generalized space $M[\Pi, \phi, q]$ naturally reduces to the classical sequence spaces studied in earlier literature. This demonstrates that our framework acts as a true generalization.

Conclusion

These results successfully generalize earlier works on Orlicz sequence spaces and contribute to the development of modular sequence space theory. By incorporating a Musielak-Orlicz modular structure, we provided a unified framework for studying coordinate-wise variable modular growth conditions.

Future Research Directions

Building upon the generalized sequence spaces introduced in this work, future investigations will focus on the following open problems:

- **Topological Duals:** Determining the exact α -, β -, and γ -duals of the spaces $l_1[\Pi, q]$ and $M[\Pi, \phi, q]$.
- **Matrix Transformations:** Characterizing the classes of infinite matrices that map from $M[\Pi, \phi, q]$ into classical spaces like l_p or c_0 .
- **Geometric Properties:** Exploring conditions under which these spaces possess strict convexity, the Banach-Saks property, or the uniform Opial property, depending on the growth sequence ϕ and the underlying seminorm q .

References

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