

INTRODUCTION AND AIM

Given a closed system composed of N economic agents with a total amount of initially distributed money M if they exchange their money x trading by pairs (u, v) in a random and local conservative way, the probability distribution $p(x)$ of the wealth of the agents tends to an exponential distribution [1]. **The decay toward this exponential equilibrium with time t can be characterized by the non-linear integral operator T given by (1) when $a = 1$.** In general, if parameter $a \neq 1$, the individual contributions to the binary exchange are au and $(2 - a)v$, then final money to be shared is $x \in (0, au + (2 - a)v)$, where local conservation is lost. This is reflected in the operator T to be integrated in the adequate region Ω .

$$p_{t+1}(x) = Tp_t(x) = \iint_{\Omega} \frac{p_t(u)p_t(v)}{au + (2 - a)v} dudv \quad (1)$$

This is the *generalized Z-model* [2] that is globally conservative, but it lacks local money conservation during binary transactions. Stability analysis, using the moments of the distribution, shows that the zeroth ($\mu_0 = 1$) and first ($\mu_1 = \langle xp(x) \rangle$) moments remain strictly constant. However, higher-order moments evolve recursively. By introducing the deviation parameter $b = 1 - a$, the evolution of the k -th moment is given by:

$$\mu_k(p_{t+1}) = \frac{1}{k+1} [((1-b)^k + (1+b)^k)\mu_k(p_t) + \text{l.o.t.}(p_t)] \quad (2)$$

For $k = 2$, this relation simplifies to:

$$\mu_2(p_{t+1}) = \frac{1}{3} [(4 - 4a + 2a^2)\mu_2(p_t) + 2a(2 - a)\mu_1^2(p_t)] \quad (3)$$

The k -th moment becomes unstable whenever $k > \ln(1/|b| + 1)/|b|$. This implies that for any non-zero deviation $|b|$, a threshold order k always exists above which all higher moments diverge. In fact, when the parameter $a \in [1 - 1/\sqrt{2}, 1 + 1/\sqrt{2}]$, moments of order $k < 3$ remain stable, while other higher-order moments can exhibit exponential instability. **Our goal is to implement a computationally stable discrete model which satisfies the previous properties of the continuous model (1).**

RESULTS AND DISCUSSION

Even though it is a simplification of the theoretical model, the total and mean wealth are conserved and when the parameter a reaches the critical value given by Eq. (3), the second moment diverges.

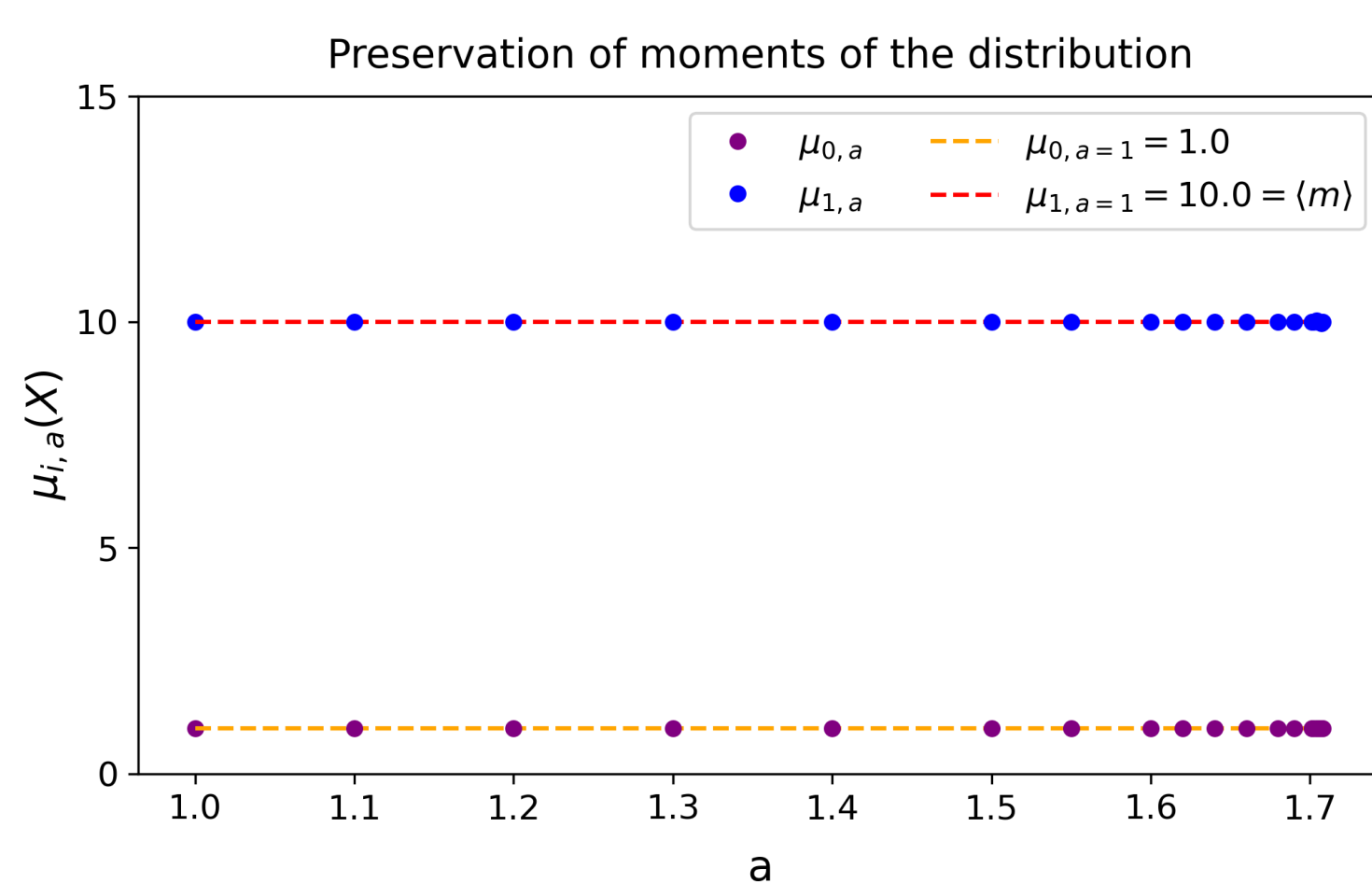


Figure 2: Preservation of the zeroth and first moments of the distribution. (Average of 50 simulations of 10^6 iterations $N = 1000$ and $M = 10000$).

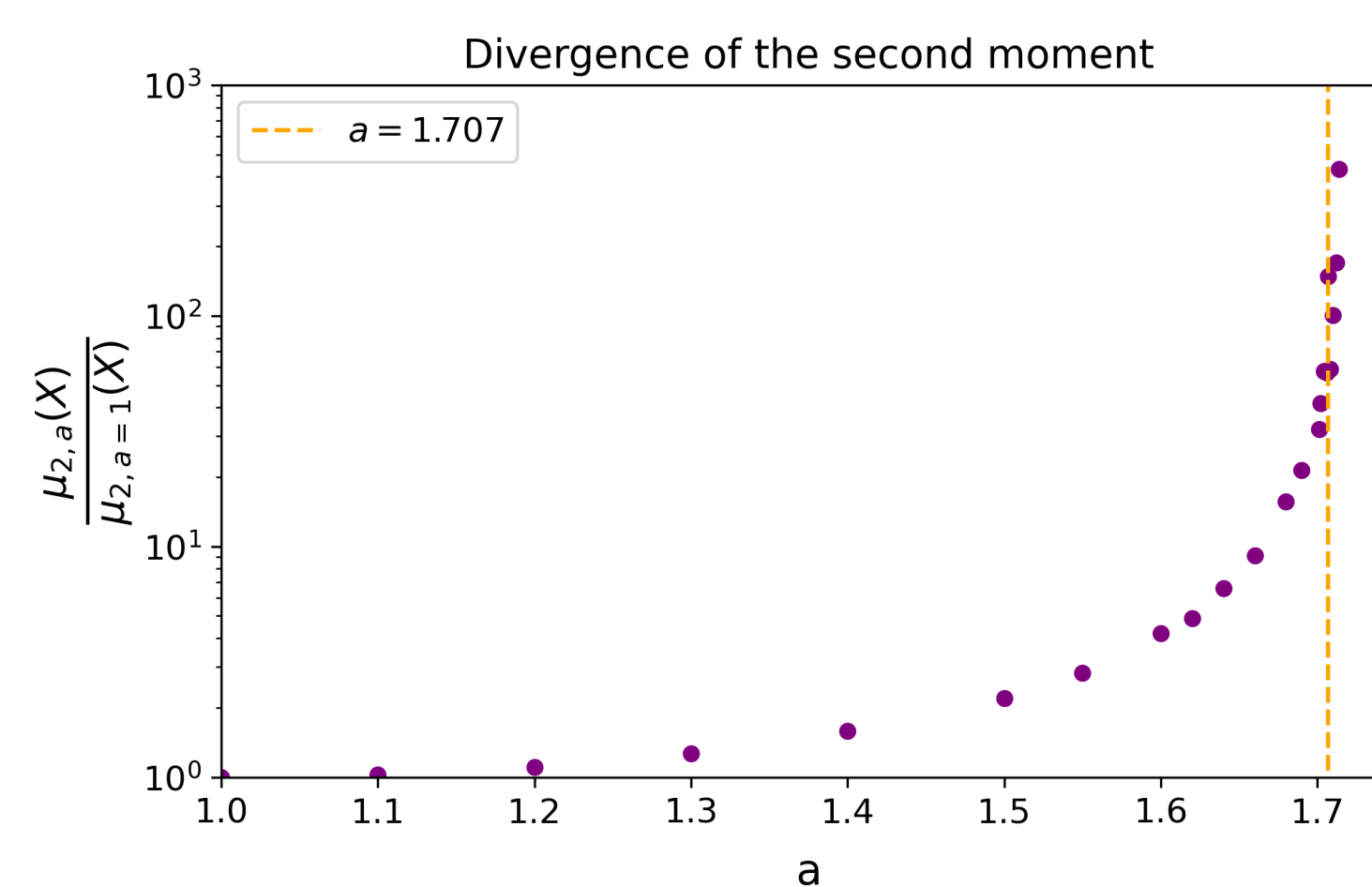


Figure 3: Divergence of the second moment of the distribution. (Average of 50 simulations of 10^6 iterations with $N = 1000$ and $M = 10000$).

The distribution tends to be symmetrical when the value of a grows, and negative money is created by the system. Since M is constant, we define the positive money as M_+ and the negative as M_- , where $M = M_+ + M_-$ must be satisfied. **We call $m^* = (M_+ + |M_-|)/M$ the monetary bubble dimension.** Figure 4 shows that the second moment diverges when the parameter tends to $a = 1.707$, therefore the system becomes completely unstable. **The computational maximum bubble dimension the system can sustain before collapsing is found to be $m^* = (|M_-| + M_+)/M = 5.18$, roughly 5 times the real money supply.**

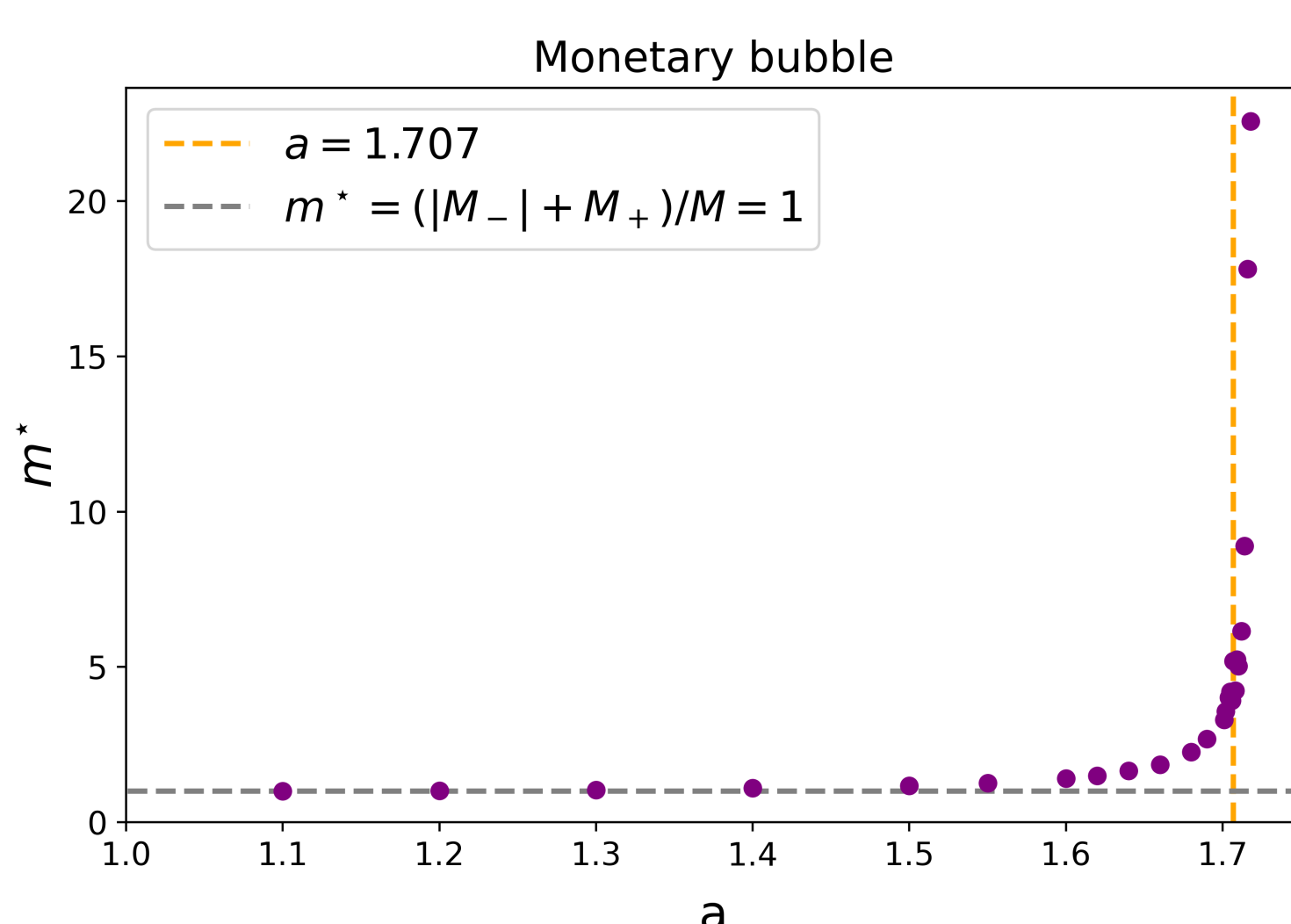


Figure 4: Analysis of the monetary bubble and its relationship with the critical value of a . (Average of 50 simulations with 10^6 iterations with $N = 1000$ and $M = 10000$).

Then, general markets can be studied assuming the monetary base M_0 corresponds with M . The generated money bubble is equivalent to “broad money” (M_2 in the US, M_4 in the UK), driven by fractional-reserve banking and highly liquid near-money. The ratio between these aggregates reflects how a country manages a money supply significantly larger than its physical base. **Figure 5 shows that, in real markets, this money ratio reached historic highs just before and during the 2008 financial crisis. Following the crash, massive central bank liquidity injections caused the ratio to collapse, stabilizing precisely around the model’s critical value of ~ 5 .**

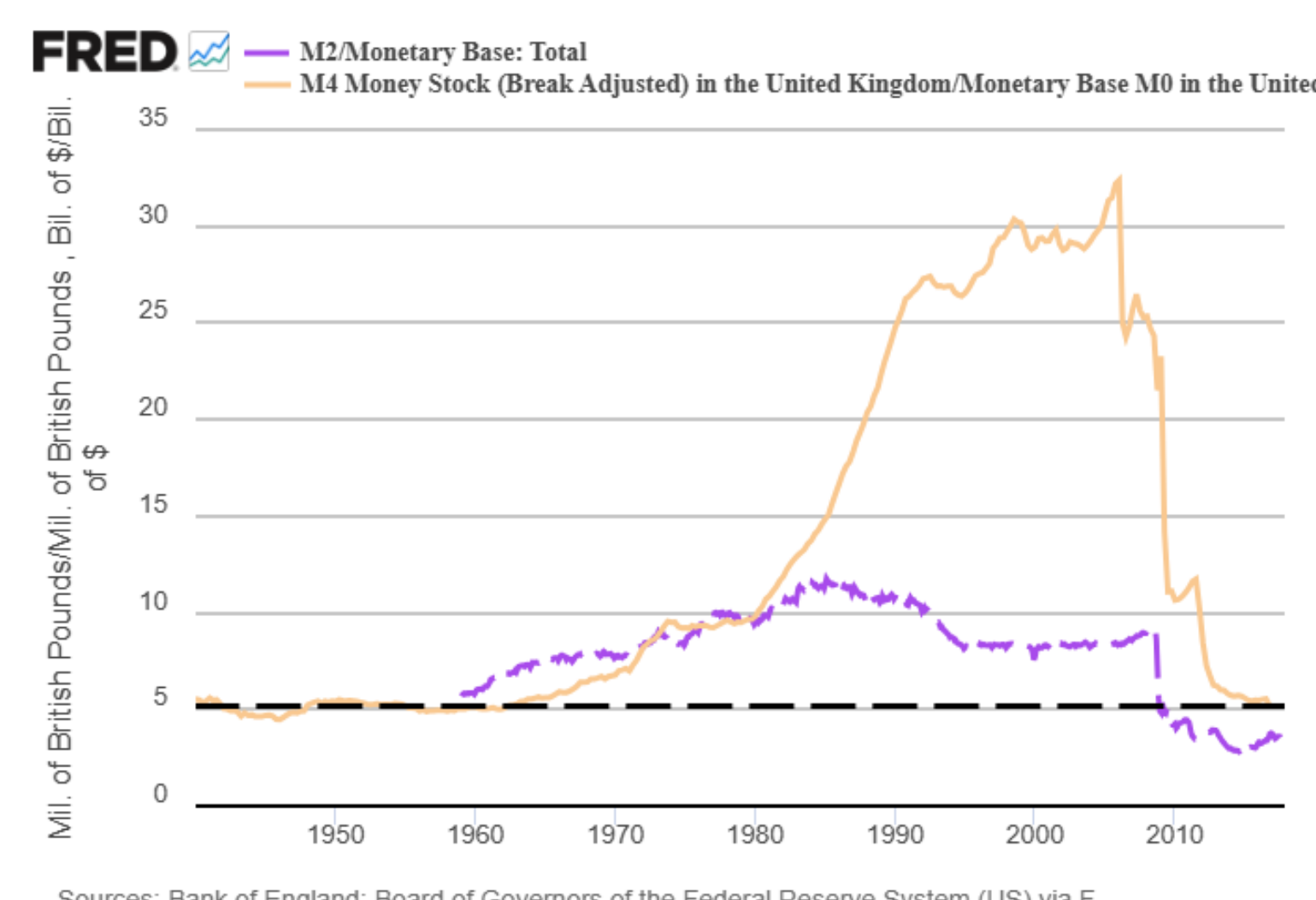


Figure 5: Evolution of M_2/M_0 in the US and M_4/M_0 in the UK. Source: FRED [3][4]. The black line corresponds to the critical value $m^* = 5.18$.

METHODS

We introduce a new discrete model with local conservation of money. The mechanism of interaction between agents is given by the equation (4), where ϵ is a random parameter.

$$\begin{cases} m'_i = (1 - a)m_i + \epsilon(am_i + (2 - a)m_j) \\ m'_j = (a - 1)m_j + (1 - \epsilon)(am_i + (2 - a)m_j) \end{cases} \quad \epsilon \in (0, 1) \quad (4)$$

We first check if there exists a steady-state distribution to which the system converges. **We observe that the stationary distribution is the Asymmetric Laplace Distribution (ALD).** The ALD is given by the equation:

$$f(x; \lambda, \kappa, \mu) = \frac{\lambda}{\kappa + 1/\kappa} \begin{cases} e^{\lambda/\kappa(x-\mu)} & \text{if } x < \mu \\ e^{-\lambda\kappa(x-\mu)} & \text{if } x \geq \mu \end{cases}$$

Widely recognized in the field of economics as it plays a key role, for example, in the study of financial risk since it is useful to model asymmetrical and sharp peaks.

Table 1 summarizes the results of fitting the simulated data to the theoretical distribution.

Variable	$a = 1.1$	$a = 1.2$	$a = 1.3$	$a = 1.4$
$\mu \pm \sigma_\mu$	0.623 ± 0.002	0.463 ± 0.071	0.245 ± 0.002	0.184 ± 0.039
$\lambda \pm \sigma_\lambda$	0.361 ± 0.001	0.277 ± 0.007	0.251 ± 0.002	0.214 ± 0.003
$\kappa \pm \sigma_\kappa$	0.273 ± 0.001	0.335 ± 0.008	0.368 ± 0.002	0.451 ± 0.006

Table 1: Parameter values for the computational fit of the simulated data to the ALD. (Average of 100 simulations of 10^6 iterations with a random initial condition)

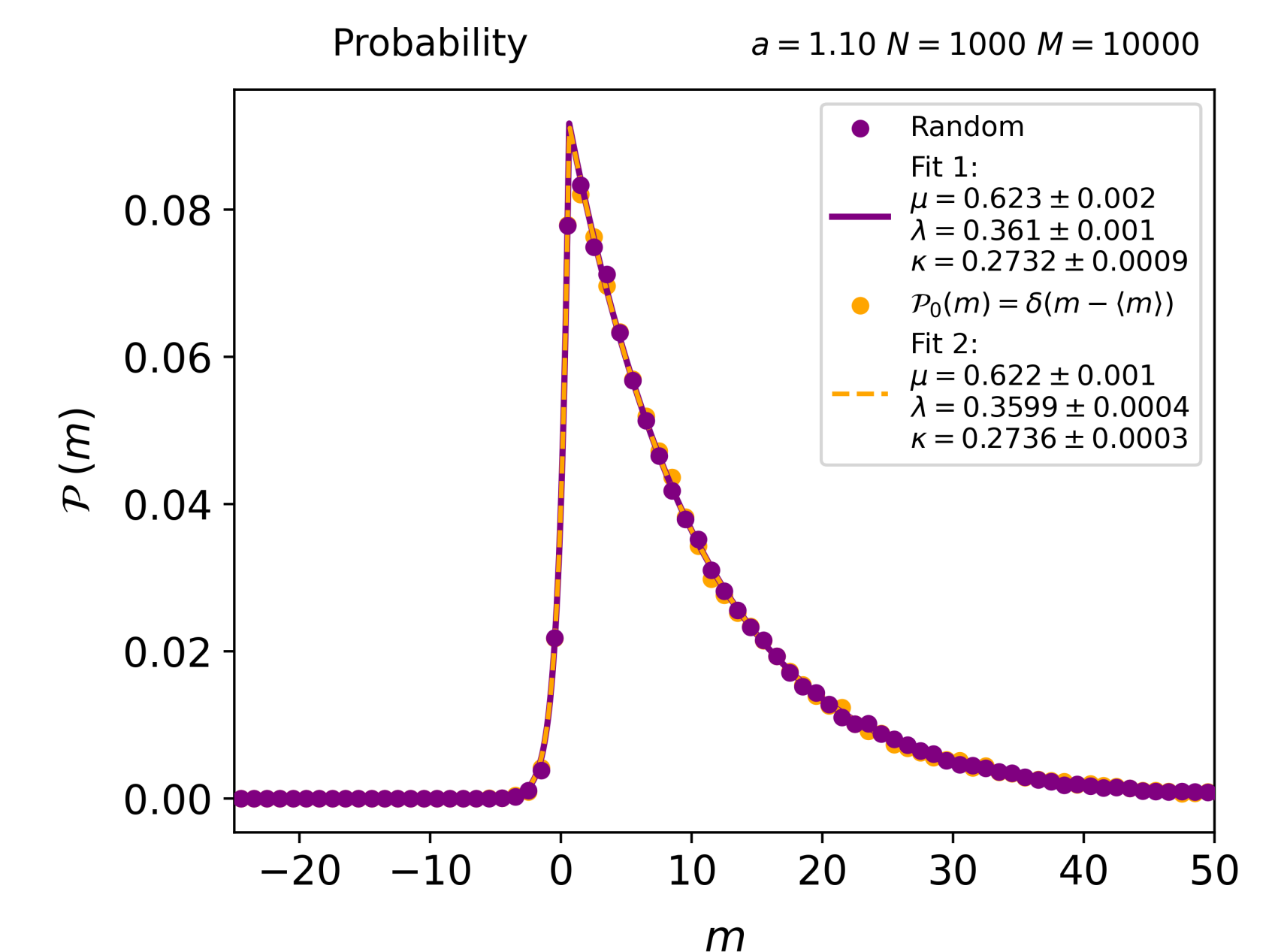


Figure 1: Fits to the ALD of the distributions obtained through simulation, with random and delta-type initial conditions (Average of 100 simulations of 10^6 iterations).

CONCLUSION

We have successfully implemented a discrete and computationally stable version of the continuous generalized Z-model. It reproduces the theoretical results and gives a powerful tool to study the stability of real markets. **While the model cannot predict the exact timing of a crisis, it successfully identifies the structural threshold where the system is entering an extremely vulnerable state.** When real markets have much more broad money than base money, the system becomes unstable as well as the simulated model. In conclusion, the formation of financial bubbles in real markets is related with the tendency of agents to accumulate a big amount of debt or loans. When the dimension of these monetary bubbles reaches the critical value of ~ 5 then the financial system becomes highly unstable giving rise to catastrophic situations as it was experienced in the 2008 global financial crisis.

FUTURE WORK

Our future work will focus on model (4), formalizing and characterizing the results concerning the Asymmetric Laplace Distribution and the formation of financial bubbles. Additionally, a new line of research emerges if this conservative model is generalized by using two parameters a and b , related with the non-conservative contributions of money, au and $(2 - b)v$, in the binary trading of each pair of agents.

REFERENCES

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