

Hexagonal Metrics & Golden-Ratio Spirals

A Metric Origin for Golden-Angle Phyllotaxis

Mahendra Samarawickrama ORCID 0000-0002-9224-6432

The Centre for Sustainable AI, Australia • Correspondence: samarawickrama@gmail.com

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Key idea. Local sixfold packing and global golden-angle growth jointly satisfy *dense local arrangement* and *non-repeating radial access*, giving the compact scaling law $r(\theta + \pi/3) = \varphi^{1/6} r(\theta)$.

INTRODUCTION & AIM

Hexagonal arrangements and golden-angle spirals are persistent motifs in natural growth, and they recur in photovoltaic solar trees, optical phased arrays, additively manufactured bone scaffolds, organoid morphogenesis and distributed sensor swarms. Each must place new units around a central source so that local neighbours fit well and global access is preserved.

Aim. Develop a metric account that explains why sixfold packing and golden-angle phyllotaxis coexist, and produce a single relation that ties them across scales.

Two coupled requirements

- **Central resource access.** Avoid persistent angular alignment so that radial shadowing is reduced.
- **Dense local packing.** Pack neighbours efficiently around each new site under an isotropic planar metric.

METHOD

We replace the implicit Euclidean ball of classical phyllotaxis with a hexagonal Minkowski norm and analyse constant-turn growth under it.

Local metric. For a regular hexagon H centred at the origin,

$$\|x\|_H = \inf\{s > 0 \mid x \in sH\}.$$

This norm is equivalent to the Euclidean norm and is invariant under rotations by $\pi/3$, encoding sixfold neighbourhood geometry.

Growth law. Place units at

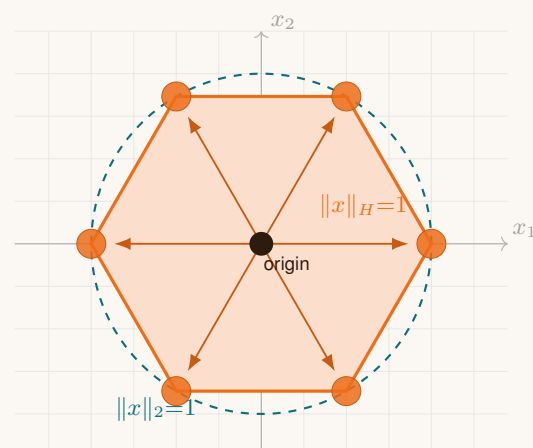
$$p_n = r_n(\cos n\alpha, \sin n\alpha), \quad r_n = r_0 \lambda^n.$$

Interpolating n gives the logarithmic spiral $r(\theta) = Ce^{k\theta}$ with $k = \ln \lambda/\alpha$.

Separation analysis. A Vogel-type area-filling model gives

$$p_n = \sqrt{n}(\cos n\alpha, \sin n\alpha), \quad \Delta_N(\alpha) = \min_{i < j} \|p_j - p_i\|_H.$$

Angle selection. Diophantine theory of badly approximable rotations selects which constant-turn rule minimises angular recurrence.



Hexagonal Minkowski unit ball (orange) versus Euclidean disk (teal dashed). Six neighbours at unit hex distance realise the sixfold local symmetry.

RESULTS & DISCUSSION

Proposition 1. Under central access and dense packing,

$$\alpha = \frac{2\pi}{\varphi^2} \approx 137.508^\circ, \quad \theta_H = \frac{\pi}{3}.$$

Their ratio $\alpha/\theta_H = 6/\varphi^2$ is irrational, so the golden angle never locks into the local hexagonal angle.

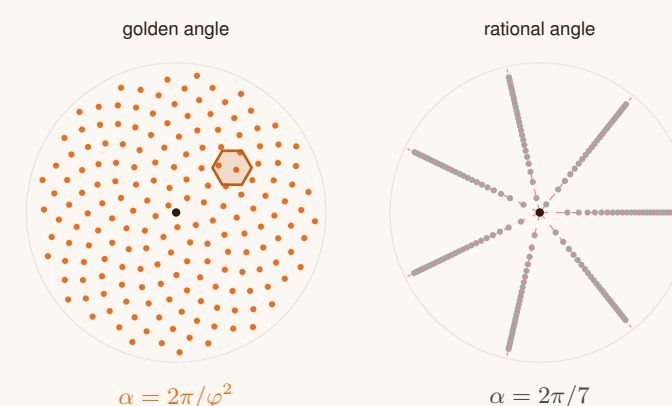
Proposition 2. If one full revolution scales radius by φ , then each hexagonal sector scales by $\varphi^{1/6}$:

$$r(\theta + \frac{\pi}{3}) = \varphi^{1/6} r(\theta), \quad k = \frac{\ln \varphi}{2\pi}.$$

Proposition 3. For badly approximable $\alpha/(2\pi)$ and $R_N = \sqrt{N}$,

$$C_1 N^{-1/2} \leq \frac{\Delta_N(\alpha)}{R_N} \leq C_2 N^{-1/2}.$$

The planar separation rate is preserved in the hexagonal norm.



Golden angle (left) distributes points without radial channels and admits sixfold local neighbourhoods (highlighted hexagon). A rational turn (right) collapses into seven radial channels (red dashed), shadowing access from the centre.

CONCLUSION

The hexagonal metric and the golden-angle spiral are not parallel coincidences. They are the local and global solutions of one two-objective problem.

- **Local.** The hexagonal Minkowski norm fixes the $\pi/3$ angle of dense planar packing.
- **Global.** The golden angle is the least resonant constant-turn rule, preserving central access.
- **Coupling.** The irrational ratio prevents resonance, and the scaling $r(\theta + \pi/3) = \varphi^{1/6} r(\theta)$ ties one full turn to six local sectors.

Design implications. The same two-objective structure carries to five engineered settings, giving a shared geometric language for systems that grow or radiate from a centre.



Organoid Solar tree Phased array Bone scaffold Sensor swarm

FUTURE WORK / REFERENCES

Predictions to test. Sixfold structure should be sharper in the hexagonal nearest-neighbour graph than in the Euclidean one. The angular power spectrum should carry components at multiples of $\pi/3$. Perturbing α toward rational multiples of 2π should create visible radial channels.

Selected references.

- [1] Vogel, H. *Math. Biosci.* **44**, 179–189 (1979). [2] Douady & Couder, *J. Theor. Biol.* **178**, 255–274 (1996). [3] Levitov, *Europhys. Lett.* **14**, 533 (1991). [4] Hales, *Discrete Comput. Geom.* **25**, 1–22 (2001). [5] Okabe, *Sci. Rep.* **5**, 15358 (2015). [6] Schneider, *Convex Bodies*, Cambridge UP (2014).