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New Insights into Hermite–Hadamard-Type Inequalities via Green Functions

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INTRODUCTION & AIM

The Hermite-Hadamard inequality is a classical inequality in convex analysis, providing lower and upper bounds for the integral average of a convex function $f:[a,b]\to\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2}$$

Its weighted form was proved by L. Fejér, and A. M. Fink discussed on its generalization (separately looking its left and right side inequality) considering certain signed measures. Using several Green functions, here we give the conditions on the real Stieltjes measure dλ, not necessarily positive, under which the Hermite-Hadamard inequality holds.

METHOD

We consider the Green functions $G_k:[\alpha,\beta]\times[\alpha,\beta]\to\mathbb{R}$, k = 0,1,2,3,4, defined by:

$$G_0(t,s)=\begin{cases}\frac{(t-\beta)(s-\alpha)}{\beta-\alpha}, & \text{for } \alpha\le s\le t,\\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha}, & \text{for } t\le s\le\beta.\end{cases}$$

$$G_1(t,s)=\begin{cases}\alpha-s, & \text{for } \alpha\le s\le t,\\ \alpha-t, & \text{for } t\le s\le\beta.\end{cases}$$

$$G_2(t,s)=\begin{cases}t-\beta, & \text{for } \alpha\le s\le t,\\ s-\beta, & \text{for } t\le s\le\beta.\end{cases}$$

$$G_3(t,s)=\begin{cases}t-\alpha, & \text{for } \alpha\le s\le t,\\ s-\alpha, & \text{for } t\le s\le\beta.\end{cases}$$

$$G_4(t,s)=\begin{cases}\beta-s, & \text{for } \alpha\le s\le t,\\ \beta-t, & \text{for } t\le s\le\beta.\end{cases}$$

Note that the functions G_k are convex under s. They are symmetric, so they are also convex under t. Also, the functions G_k are continuous under s and continuous under t.

The following lemma holds:

Lemma

For every function $\varphi:[\alpha,\beta]\to\mathbb{R},\ \varphi\in C^2([\alpha,\beta])$, the following identities hold:

$$\varphi(x)=\frac{\beta-x}{\beta-\alpha}\varphi(\alpha)+\frac{x-\alpha}{\beta-\alpha}\varphi(\beta)+\int_\alpha^\beta G_0(x,s)\varphi''(s)ds,$$

$$\varphi(x)=\varphi(\alpha)+(x-\alpha)\varphi'(\beta)+\int_\alpha^\beta G_1(x,s)\varphi''(s)ds,$$

$$\varphi(x)=\varphi(\beta)+(x-\beta)\varphi'(\alpha)+\int_\alpha^\beta G_2(x,s)\varphi''(s)ds,$$

$$\varphi(x)=\varphi(\beta)-(\beta-\alpha)\varphi'(\beta)+(x-\alpha)\varphi'(\alpha)+\int_\alpha^\beta G_3(x,s)\varphi''(s)ds,$$

$$\varphi(x)=\varphi(\alpha)+(\beta-\alpha)\varphi'(\alpha)-(\beta-x)\varphi'(\beta)+\int_\alpha^\beta G_4(x,s)\varphi''(s)ds.$$

After some calculation, one can easily get that for all functions G_k the following identity holds, and that identity will be crucial for deriving further results:

$$\frac{\int_a^b\varphi(g(x))\,d\lambda(x)}{\int_a^b d\lambda(x)}-\varphi(\overline{g})=\int_\alpha^\beta\left[\frac{\int_a^b G_k(g(x),s)\,d\lambda(x)}{\int_a^b d\lambda(x)}-G_k(\overline{g},s)\right]\varphi''(s)ds$$

RESULTS & DISCUSSION

Theorem 1

Let $g:[a,b]\to\mathbb{R}$ be continuous function and $[\alpha,\beta]$ an interval such that the image of g is a subset of $[\alpha,\beta]$. Let $\lambda:[a,b]\to\mathbb{R}$ be continuous function or the function of bounded variation, such that $\lambda(a)\neq\lambda(b)$ and $\overline{g}\in[\alpha,\beta]$, where $\overline{g}=\frac{\int_a^b g(x)d\lambda(x)}{\int_a^b d\lambda(x)}$

Then the following two statements are equivalent:

(1) For every continuous convex function $\varphi:[\alpha,\beta]\to\mathbb{R}$ holds:

$$\varphi(\overline{g})\le\frac{\int_a^b\varphi(g(x))\,d\lambda(x)}{\int_a^b d\lambda(x)}$$

(2) For all $s\in[\alpha,\beta]$ holds

$$G_k(\overline{g},s)\le\frac{\int_a^b G_k(g(x),s)\,d\lambda(x)}{\int_a^b d\lambda(x)}.$$

Theorem 2

Let $g:[a,b]\to\mathbb{R}$ be continuous function and $[\alpha,\beta]$ an interval such that the image of g is a subset of $[\alpha,\beta]$. Let $m,M\in[\alpha,\beta]\,(m\neq M)$ be such that $m\le g(t)\le M$ for all $t\in[a,b]$. Let $\lambda:[a,b]\to\mathbb{R}$ be continuous function or the function of bounded variation, and $\lambda(a)\neq\lambda(b)$. Then the following two statements are equivalent:

(1) For every continuous convex function $\varphi:[\alpha,\beta]\to\mathbb{R}$ holds

$$\frac{\int_a^b\varphi(g(x))\,d\lambda(x)}{\int_a^b d\lambda(x)}\le\frac{M-\overline{g}}{M-m}\varphi(m)+\frac{\overline{g}-m}{M-m}\varphi(M).$$

(2) For all $s\in[\alpha,\beta]$ holds:

$$\frac{\int_a^b G_k(g(x),s)\,d\lambda(x)}{\int_a^b d\lambda(x)}\le\frac{M-\overline{g}}{M-m}G_k(m,s)+\frac{\overline{g}-m}{M-m}G_k(M,s).$$

Note: If we set in $m=\alpha$ and $M=\beta$, the inequality in (2) transforms into

$$\frac{\int_a^b G_k(g(x),s)\,d\lambda(x)}{\int_a^b d\lambda(x)}\le 0.$$

Corollary 1

Let $\lambda:[a,b]\to\mathbb{R}$ be continuous function or the function of bounded variation and $\lambda(a)\neq\lambda(b)$. Let $[\alpha,\beta]\subseteq\mathbb{R}$ be such that $[a,b]\subseteq[\alpha,\beta]$ and $\overline{x}\in[\alpha,\beta]$ where $\overline{x}=\frac{\int_a^b x\,d\lambda(x)}{\int_a^b d\lambda(x)}$.
Then the following two statements are equivalent:

(1) For every continuous convex function $f:[\alpha,\beta]\to\mathbb{R}$ holds

$$f(\overline{x})\le\frac{\int_a^b f(x)\,d\lambda(x)}{\int_a^b d\lambda(x)}.$$

(2) For all $s\in[\alpha,\beta]$ holds

$$G_k(\overline{x},s)\le\frac{\int_a^b G_k(x,s)\,d\lambda(x)}{\int_a^b d\lambda(x)}.$$

Note: For $\lambda(x)=x$, it is $\int_a^b d\lambda(x)=b-a$ and $\overline{x}=\frac{a+b}{2}$, so the inequality in (1) becomes the left inequality in the classical Hermite-Hadamard inequality.

Corollary 2

Let $\lambda:[a,b]\to\mathbb{R}$ be continuous function or the function of bounded variation and $\lambda(a)\neq\lambda(b)$.
Let $[\alpha,\beta]\subseteq\mathbb{R}$ be such that $[a,b]\subseteq[\alpha,\beta]$.
Then the following two statements are equivalent:

(1) For every continuous convex function $f:[\alpha,\beta]\to\mathbb{R}$ holds

$$\frac{\int_a^b f(x)\,d\lambda(x)}{\int_a^b d\lambda(x)}\le\frac{b-\overline{x}}{b-a}f(a)+\frac{\overline{x}-a}{b-a}f(b).$$

(2) For all $s\in[\alpha,\beta]$ holds

$$\frac{\int_a^b G_k(x,s)\,d\lambda(x)}{\int_a^b d\lambda(x)}\le\frac{b-\overline{x}}{b-a}G_k(a,s)+\frac{\overline{x}-a}{b-a}G_k(b,s).$$

Note: Consider the case when k=0. Setting $\alpha=a$ and $\beta=b$ in previous corollary, we get that the right side of the inequality in (2) equals to zero, so that inequality becomes

$$\frac{\int_a^b G_0(x,s)\,d\lambda(x)}{\int_a^b d\lambda(x)}\le 0.$$

Note: For $\lambda(x)=x$ inequality in (1) becomes the right-side inequality in the classical Hermite-Hadamard inequality.