

Critical Anisotropic Quasilinear Equations with External Potentials in \mathbb{R}^n

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INTRODUCTION & AIM

Nonlinear elliptic equations involving critical Sobolev exponents play an important role in nonlinear analysis and mathematical physics. In the classical setting, such problems are associated with the standard p -Laplacian operator and Euclidean geometry. Recently, anisotropic models described by Finsler norms have attracted increasing attention, as they provide a natural framework for studying direction-dependent diffusion phenomena.

In this work, we investigate an anisotropic quasilinear problem with critical growth and an external potential. The main objective is to establish existence and multiplicity results using variational methods and to analyze the influence of anisotropy on the compactness properties and qualitative behavior of weak solutions. More precisely, we consider the following problem:

$$-div(H(\nabla u)^{p-1} \nabla H(\nabla u)) = |u|^{p^*-2} u + \lambda f(x), x \in \mathbb{R}^n$$

where $1 < p < n$, $p^* = np/(n-p)$ is the critical Sobolev exponent, and (H) is a uniformly convex Finsler norm.

METHOD

The problem is studied within a variational framework in the anisotropic Sobolev space associated with the Finsler norm. The analysis relies mainly on modern tools from nonlinear functional analysis and critical point theory. In particular, the Ekeland variational principle and the Mountain Pass Theorem are used to establish the existence of nontrivial weak solutions. The study also requires a careful Palais–Smale analysis together with anisotropic Sobolev inequalities in order to overcome the lack of compactness caused by the critical Sobolev exponent and the unbounded domain \mathbb{R}^n .

Main difficulty:

The main difficulty comes from the lack of compactness caused by the critical Sobolev exponent in the unbounded domain \mathbb{R}^n . In this case, minimizing sequences may fail to converge strongly, which makes the application of standard variational methods more delicate.

RESULTS & DISCUSSION

RESULTS:

The obtained results show that, under suitable assumptions on the parameter λ , the external term f and the Finsler norm H , the anisotropic critical problem admits at least two nontrivial weak solutions obtained by variational methods. The analysis proves that the variational structure is preserved in the anisotropic framework despite the lack of compactness caused by the critical Sobolev exponent.

Discussion:

The presence of the Finsler norm modifies the diffusion behavior of the equation and affects the compactness properties of the associated variational problem. In particular, the anisotropic structure leads to directional effects that do not appear in the classical Euclidean framework. These results show that variational methods can still be applied successfully in the anisotropic setting.

CONCLUSION

This work shows that critical anisotropic quasilinear equations involving Finsler norms admit nontrivial weak solutions despite the presence of critical growth and loss of compactness. The variational approach remains effective in the anisotropic framework. Moreover, the anisotropic structure significantly influences the behavior of solutions and leads to directional concentration phenomena compared to the classical Euclidean case.

FUTURE WORK / REFERENCES

Future Work:

Future research directions include Kirchhoff-type anisotropic problems, multiplicity and qualitative properties of solutions, and extensions to fractional and variable exponent settings.

References:

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