IOCMA 2026 Conference
The 2nd International Online Conference on Mathematics and Applications
10–12 June 2026 | Online
MDPI

Critical Points of Quotients of Finite Blaschke Products of Small Degrees

N.Kajan[1,2], K.M.N.M. Chathuranga[3], P.G.R.S Ranasinghe[3]
[1]Department of Physical Science, Faculty of Applied Sciences, University of Vavuniya, Sri Lanka.
[2]Postgraduate Institute of Science (PGIS), University of Peradeniya, Sri Lanka
[3]Department of Mathematics, Faculty of Science, University of Peradeniya, Sri Lanka

INTRODUCTION & AIM

Finite Blaschke products are important class of Meromorphic function on $\mathbb{C}$ characteristic by following three properties. They map $\mathbb{D}$ to $\mathbb{D}$ holomorphically, continuously extended to unit circle $\mathbb{T}$, and they map unit circle onto itself. It has been shown that a Blaschke Product itself finite product of Möbius transformation and they are fundamental in complex analysis and have been studied extensively. Even though quotient of finite Blaschke product, also have many interesting properties. They have not been investigated as much as their counter part. In fact, in literature there is no study on the distribution of critical points of such functions. This study focuses on critical points of quotient of Blaschke products of degree two and three.

It is relatively easy exercises to prove that such a function with a zero in $\mathbb{D}$ can be reduced to the form $B(z) = z\left(\frac{z-a}{1-\bar{a}z}\right)\left(\frac{z-b}{1-\bar{b}z}\right)$; , where $|a||b| \neq 1$. When $a, b \in \mathbb{D}$, this is indeed a Blaschke Product, and it is well known that they do not have any critical points in $\mathbb{D}$. However, the study proves that situation is vastly different for quotient of Blaschke products.

METHOD

A quotient of finite Blaschke product can be expressed in the form

$$B(z) = e^{i\theta} \prod_{k=1}^{n} \left(\frac{z-a_k}{1-\overline{a_k}z}\right);$$

for some $n \in \mathbb{Z}^+$, for all $k = 1,2, \ldots \ldots, n$, and some $\theta \in [0,2\pi)$, $a_k \notin \mathbb{T}$.

In general, B is meromorphic in $\mathbb{D}$, continuous on to $\mathbb{T}$, and maps $\mathbb{T}$ to itself. If $a_k \in \mathbb{D}$, for all $k = 1,2, \ldots \ldots, n$, then B is exactly a finite Blaschke product of degree n.

It is well known that finite Blaschke products have no critical points on $\mathbb{T}$. To see this one can observe that

$$z\frac{B'(z)}{B(z)} = \sum_{k=1}^{n} \frac{1-|a_k|^2}{|z-a_k|^2},$$

for all $z \in \mathbb{T}$. When $|a_k|<1$, for k, $z\frac{B'(z)}{B(z)}$ is a strictly positive real number and can not be zero.

However at least one k value such that $|a_k| > 1$. This prove does not applies. In fact , in our study we show the existence of critical points on $\mathbb{T}$ for such function for $n = 2$ and 3.

RESULTS & DISCUSSION

Consider the simplest case of $n = 2$,

$$B_1(z) = z\left(\frac{z-a}{1-\bar{a}z}\right);\ |a| > 1,$$

We can show that this function has two critical points given by the intersection of two orthogonal circles $|z| = 1$ and $|z - a| = \sqrt{|a|^2 - 1}$. These two circles happened to be the pre-image of $\mathbb{T}$ under B_1.

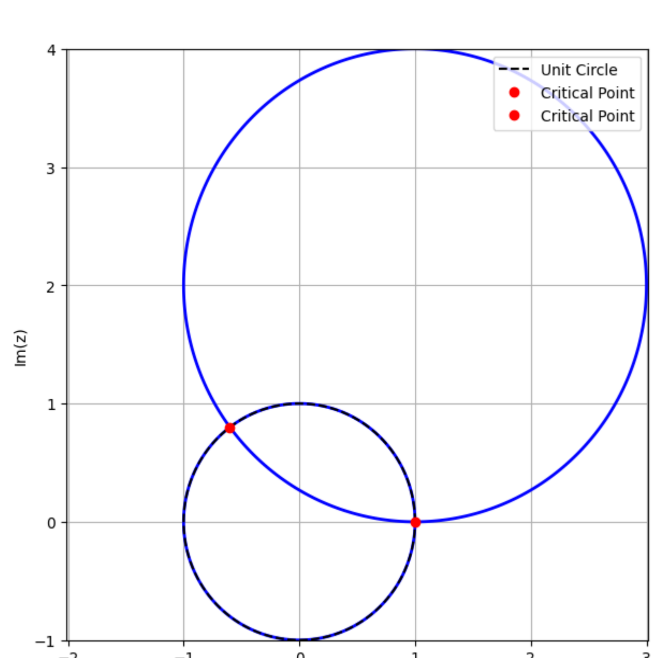

Figure 1 : Unit circle, critical points, and level curve |B(z)|=1 for a=1+2i.

In the case of $n = 3$,

$$B_2(z) = z\left(\frac{z-a}{1-\bar{a}z}\right)\left(\frac{z-b}{1-\bar{b}z}\right); |a|, |b| \neq 1$$

At least one of a or b is not in $\mathbb{D}$, the critical points on $\mathbb{T}$ also satisfy fourth degree polynomial

$\bar{a}\bar{b}\ z^4 - 2(\bar{a} + \bar{b})z^3 + (3 + |a + b|^2 - |ab|^2)z^2 - 2(a + b)z + ab = 0$. These critical points are exactly the intersection of $|z| = 1$ and

$\left((|z - a|^2 - |a|^2 + 1)(|z - b|^2 - |b|^2 + 1) - |z|^2(1 + |a|^2)(1 + |b|^2)\right) = 0.$

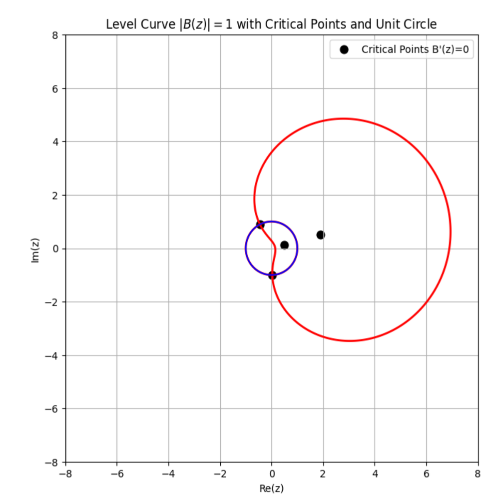

Figure 2: Unit circle, critical points, and level curve |B(z)| = 1 for a = 2 + i,b = −1.5+2i

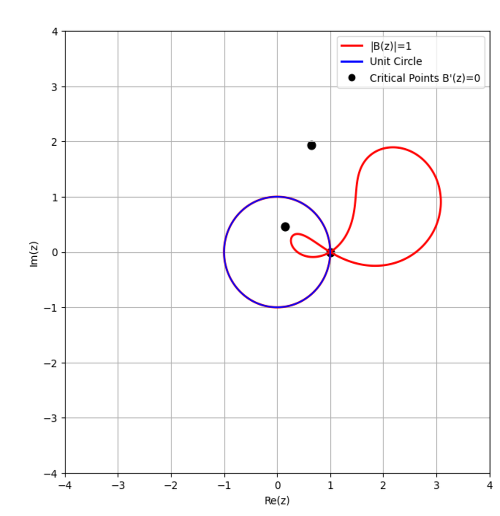

Figure 3: Unit circle, critical points, and level curve |B(z)| = 1 for a = 2 +i,b=0.5+ 0.5i

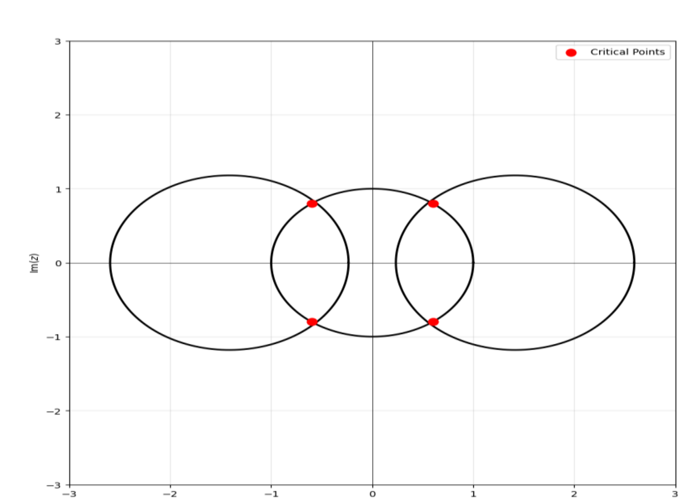

Figure 4 : Unit circle, critical points, and level curve |B(z)|=1 for a=−b = $\sqrt{2}$

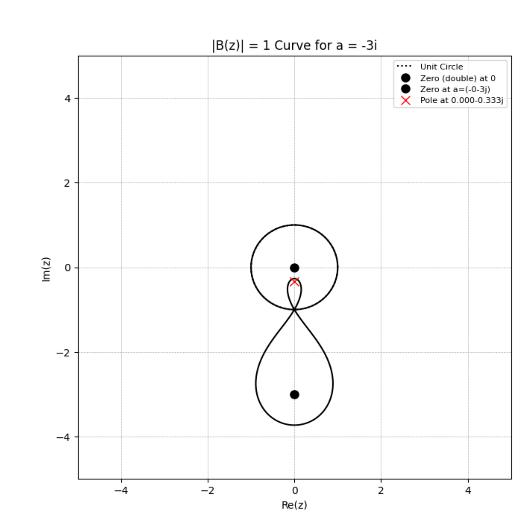

Figure 5: Unit circle, critical points, and level curve |B(z)|=1 for a=−3i

CONCLUSION

The study demonstrates that quotients of finite Blaschke products exhibit fundamentally different critical point behavior from finite Blaschke products.

In particular, critical points may occur on the unit circle, and their locations are closely related to the geometry of the level curve ($|B(z)| = 1$). The results obtained for low degree cases provide a geometric characterization of these critical points and suggest several directions for future research on higher-degree quotient Blaschke products.

REFERENCES

[1] Blaschke, W. (1929). Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie. Springer.
[2] Gamelin, T. W. (2001). Complex analysis. Springer.
[3] Garnett, J. B. (2007). Bounded analytic functions (Rev. ed.). Springer.
[4] Milnor, J. (2006). Dynamics in one complex variable (3rd ed.). Princeton University Press.
[5] Pommerenke, C. (1992). Boundary behaviour of conformal maps. Springer.
[6] Singer, D. (1996). Foci of conics and the critical points of Blaschke products. Proceedings of the American Mathematical Society, 124(12), 3643–3652.
[7] Zakeri, S. (2005). Dynamics of cubic Blaschke products. Ergodic Theory and Dynamical Systems, 25(3), 971–1009.