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Mild Solution Existence for Nonlinear φ-Caputo Fractional Differential Equation in Banach Space

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INTRODUCTION & AIM

Fractional differential equations involving the φ-Caputo derivative are powerful tools for modelling processes with **memory** and **hereditary properties**. We investigate a nonlinear class of such equations in a real Banach space $(X, \|\cdot\|)$.

Initial Value Problem (IVP):

$$\begin{cases} {}^{C}D_{a+}^{\alpha,\varphi} x(t) + \omega\, {}^{C}D_{a+}^{\alpha-1,\varphi} g(t, x(t)) = f(t, x(t)), & t \in I = [a, b], \\ x(a) = x'(a) = 0, \end{cases} \tag{1}$$

where $\omega > 0$, $\alpha \in (1, 2)$, ${}^{C}D_{a+}^{\alpha,\varphi}$ is the φ-Caputo derivative of order $\alpha \in \{\alpha, \alpha - 1\}$, $\varphi \in \mathcal{S}(I, \mathbb{R})$ (i.e. $\varphi' > 0$), and $f, g \in C(I \times X; X)$.

Research Objectives:

- Establish **existence** of mild solutions via the Meir–Keeler fixed-point theorem combined with the **measure of noncompactness (MNC)**.
- Prove **uniqueness** via the Banach contraction principle.
- Analyse **sensitivity** to initial data.
- Examine **Ulam–Hyers stability** of solutions.

METHOD

Mild Solution. A function $x \in C(I; X)$ is a *mild solution* of (1) if

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(\alpha)} \int_a^t \tilde{\varphi}(t, s)^{\alpha-1} \varphi'(s)\, f(s, x(s))\, ds \\ & - \omega \int_a^t \varphi'(s)\, g(s, x(s))\, ds + \omega\, g(a, 0)\, \tilde{\varphi}(t, a), \end{aligned} \tag{2}$$

where $\tilde{\varphi}(t, s) = \varphi(t) - \varphi(s)$.

Fixed-Point Operator. Define $\mathcal{T} : C(I; X) \to C(I; X)$ by the right-hand side above. Fixed points of $\mathcal{T}$ are exactly the mild solutions of (1).

Key Assumptions:

(C1) $\|f(t, x)\| \le f_1(t) + f_2(t)\, \|x\|$ and $\|g(t, x)\| \le g_1(t) + g_2(t)\, \|x\|$ for all $x \in X$.

(C2) $\mathcal{T}$ maps the closed ball $B_R \subset C(I; X)$ into itself for some $R > 0$.

(C3) $\mu(f(t, A)) \le \beta_f(t)\, \mu(A)$ and $\mu(g(t, A)) \le \beta_g(t)\, \mu(A)$ for every bounded $A \subset X$.

(C4) $\|f(t, x_1) - f(t, x_2)\| \le L_f\, \|x_1 - x_2\|$ and $\|g(t, x_1) - g(t, x_2)\| \le L_g\, \|x_1 - x_2\|$.

RESULTS & DISCUSSION

Theorem 2.1 (Existence — Meir–Keeler)

Under **(C1)**, **(C2)**, **(C3)**, problem (1) admits **at least one mild solution** $x \in B_R \subset C(I; X)$.

Illustrative Example (Theorem 2.1 applied):

Let $X = \ell^1(\mathbb{R})$, $\alpha = \frac{3}{2}$, $I = [1, 2]$, $\omega = \frac{1}{2}$, $\varphi_1(t) = t$, $\varphi_2(t) = \ln t$, and

$$f(t, x) = \left(\frac{1}{t+3} \left(\frac{1}{2^n} + x_n \right) \right)_{n \ge 1}, \qquad g(t, x) = \left(\frac{1}{\ln t + 2} \left(\frac{1}{3^n} + \sin |x_n| \right) \right)_{n \ge 1}.$$

The Hausdorff MNC in X is $\mu(A) = \lim_{j \to \infty} \sup_{x \in A} \sum_{n \ge j} |x_n|$. All conditions **(C1)–(C3)** are verified; hence (1) admits **at least one mild solution** by Theorem 2.1.

CONCLUSION

This work advances the study of φ-Caputo fractional differential equations in Banach spaces by providing:

- **Existence** of mild solutions via the Meir–Keeler theorem and MNC.
- **Uniqueness** via the Banach contraction principle.
- **Ulam–Hyers stability**, ensuring solution robustness under perturbations.

The results form a rigorous foundation for further investigation of fractional models with generalised memory effects.

FUTURE WORK / REFERENCES

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