

Abstract

In this work a Henstock-Kurzweil integral is defined on a measure space (X, μ, \mathcal{A}) using fractal structures. Furthermore, some properties and results like *Cauchy's Criterion*, *additivity* and *Henstock's Lemma* are shown, as well as "limit theorems" such us *Monotone* and *Dominated Convergence theorems*. Finally, the *Fundamental Theorem of Calculus* is presented in the context of Linearly Ordered Topological Spaces (LOTS).

Fractal structures

Definition. A *fractal structure* is a countable family of coverings $\Gamma = \{\Gamma_n : n \in \mathbb{N}\}$ such that every element of Γ_{n+1} is contained in some element of Γ_n and for each $B \in \Gamma_n$ then $B = \cup\{A \in \Gamma_{n+1} : A \subseteq B\}$. These two conditions are gathered by the notation $\Gamma_{n+1} \prec \prec \Gamma_n$ and the cover Γ_n is said to be the *level n of the fractal structure*.

Let (X, \mathcal{A}, μ) be a measure space. These are some interesting properties of fractal structures:

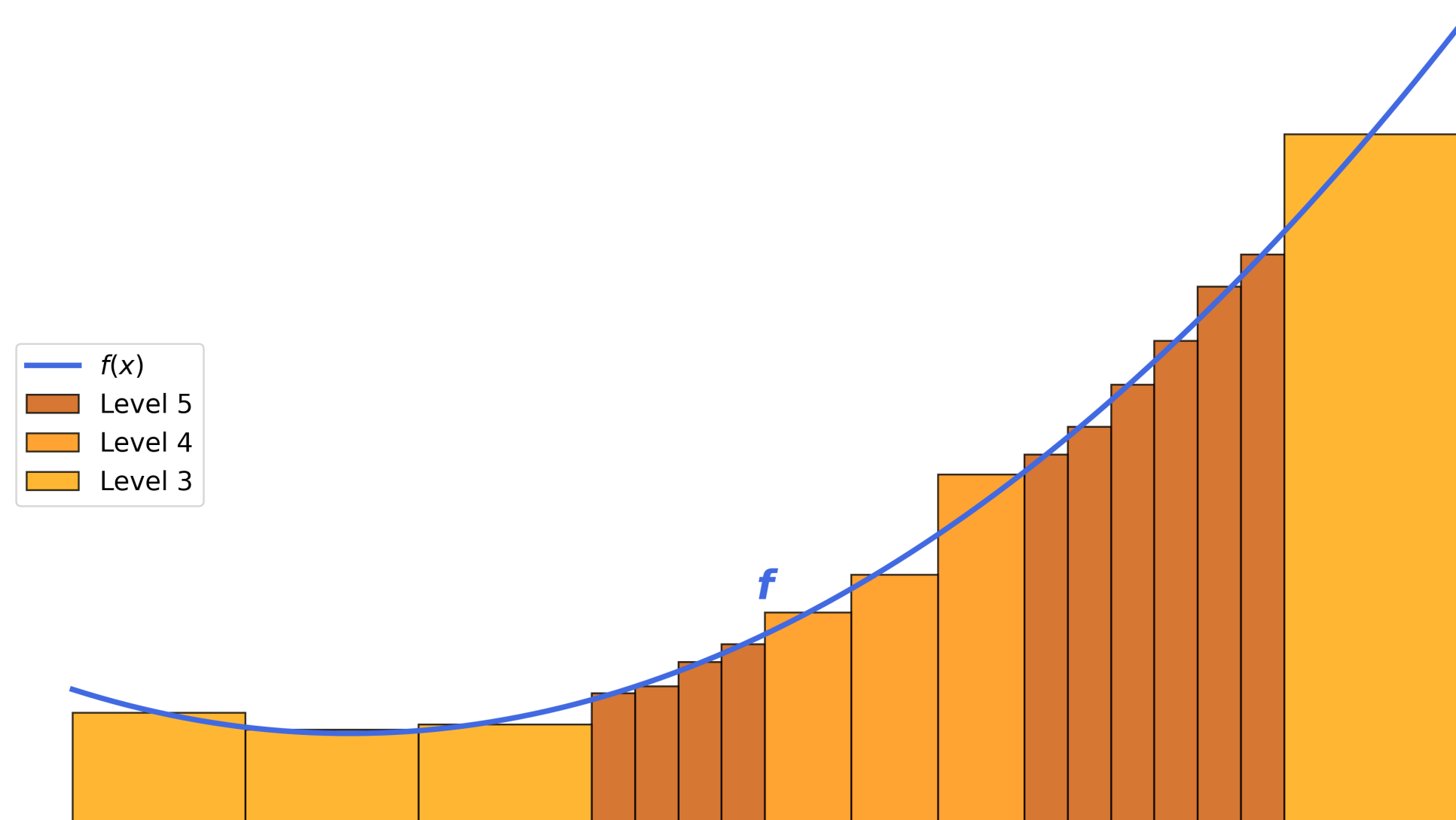
1. Γ is μ -disjoint if $\Gamma_n \subseteq \mathcal{A}$ is countable for each $n \in \mathbb{N}$ and $\mu(A \cap B) = 0$ if $A, B \in \Gamma_n$ are different elements for all $n \in \mathbb{N}$.
2. Γ is a *finite* fractal structure if every level Γ_n of Γ is a finite covering of X .
3. Γ is *Cantor-complete* if for each decreasing sequence $(A_n)_{n \in \mathbb{N}}$ ($A_{n+1} \subseteq A_n$, for each $n \in \mathbb{N}$), with $A_n \in \Gamma_n$, $\cap_{n \in \mathbb{N}} A_n$ is not empty.
4. A fractal structure $\Gamma = \{\Gamma_n : n \in \mathbb{N}\}$ is a *tiling* if Γ_n is a tiling for each $n \in \mathbb{N}$, that is, $A = \overline{A}$ for all $A \in \Gamma_n$ and for each $A, B \in \Gamma_n$ with $A \neq B$ then $A^\circ \cap B^\circ = \emptyset$.
5. Γ is *starbase* if $\{St(x, \Gamma_n) : n \in \mathbb{N}\}$ is a neighbourhood basis for x , where $St(x, \Gamma_n) = \cup\{A \in \Gamma_n : x \in A\}$.

HK-fractal-integral in a measure space

Definition. Let (X, μ, \mathcal{A}) be a measure space and a fractal structure Γ . A *tagged partition* is a finite collection $\mathcal{D} = \{(x, A, n)\}$ satisfying that $x \in A \in \Gamma_n$, $\mu(A \cap B) = 0$, for every distinct A and B in \mathcal{D} , and $X = \cup_{\mathcal{D}} A$. In addition, a *gauge* is a function $\gamma : X \rightarrow \mathbb{N}$ and given a tagged partition $\mathcal{D} = \{(x, A, n)\}$, it is γ -fine if $n \geq \gamma(x)$, for each $(x, A, n) \in \mathcal{D}$.

Definition. A fractal structure Γ is a *gauge-compatible* (GC-)fractal structure if it is μ -disjoint, finite and Cantor-complete. This property ensures the existence of γ -fine tagged partitions.

Definition. Let $f : X \rightarrow \mathbb{R}$ be a function and Γ a GC-fractal structure. We say that f is HK-integrable with respect to μ and Γ , denoted by $f \in \mathcal{HK}(X, \mu, \Gamma)$ if there exists $I \in \mathbb{R}$ such that for every $\varepsilon > 0$, there is a gauge $\gamma : X \rightarrow \mathbb{N}$ such that $|S(f, \mu, \mathcal{D}) - I| < \varepsilon$, for every γ -fine partition $\mathcal{D} = \{(x, A, n)\}$. The term $S(f, \mu, \mathcal{D}) = \sum_{\mathcal{D}} f(x)\mu(A)$ is called the *Riemann sum* of f with respect to μ and \mathcal{D} and the value of the integral is denoted by $I = (\Gamma) \int_X f d\mu$.



The previous figure shows a real-valued function being integrated using the natural fractal structure on an interval. This fractal structure, denoted by Γ_{nat} is defined in the next way: let $J = [a, b]$ be a compact interval in \mathbb{R} and $\Gamma_n = \left\{ \left[\frac{k(b-a)}{2^n} + a, \frac{(k+1)(b-a)}{2^n} + a \right] : k \in \{0, \dots, 2^n - 1\} \right\}$, then $\Gamma_{\text{nat}} = \{\Gamma_n : n \in \mathbb{N}\}$.

Authors have proved that a similar result to **Cauchy's Criterion** is obtained for this integral, and by supposing the tiling condition on a GC-fractal structure, the integral becomes an **additive operator**. The last property yields a result equivalent to **Henstock's Lemma**, ensuring that, for a fixed $\varepsilon > 0$, Riemann sum approximations for partial partitions are nearly as accurate as those for partitions that covers the entire space.

Convergence Theorems

Some classical convergence theorems still hold for this integral. Let $(f_k)_{k \in \mathbb{N}}$ be a sequence of functions such that $f_k \rightarrow f$ and $f_k \in \mathcal{HK}(X, \mu, \Gamma)$ and let $g \in \mathcal{HK}(X, \mu, \Gamma)$ be non-negative. Suppose that $\mu(X) < \infty$.

• **Monotone Convergence theorem.** If $f_k(x) \leq f_{k+1}(x)$ or $f_k(x) \geq f_{k+1}(x)$ for each $x \in X$ and $k \in \mathbb{N}$, then $f \in \mathcal{HK}(X, \mu, \Gamma)$ if and only if $\lim_{k \rightarrow \infty} (\Gamma) \int_X f_k d\mu < \infty$.

• **Dominated Convergence theorem.** If $|f_k(x)| \leq g(x)$ for all $x \in X$ and $k \in \mathbb{N}$, then $f \in \mathcal{HK}(X, \mu, \Gamma)$.

If any of both theorems holds, we have that

$$(\Gamma) \int_X \lim_{k \rightarrow \infty} f_k d\mu = (\Gamma) \int_X f d\mu = \lim_{k \rightarrow \infty} (\Gamma) \int_X f_k d\mu.$$

Linearly Ordered Topological Spaces

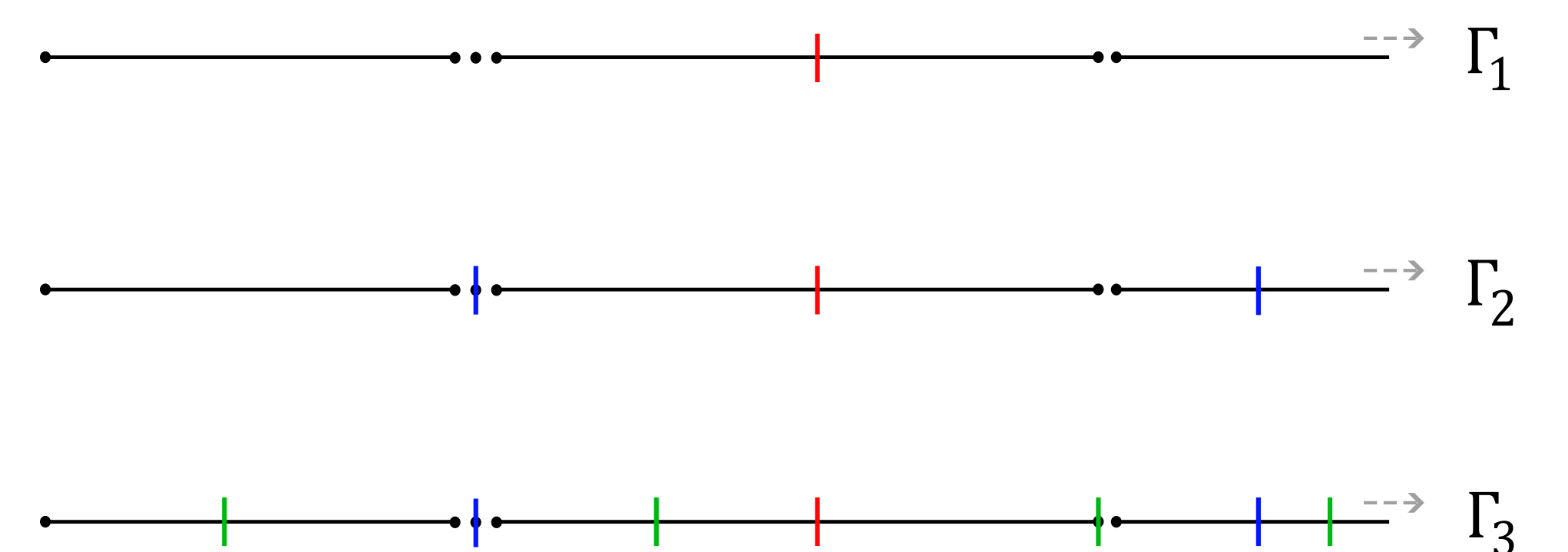
Definition. A *Linearly Ordered Topological Space* (LOTS) is a linearly or totally ordered set X with the order topology, i.e, the topology generated by the subbasis $\{\{x \in X : x < a\} : a \in X\} \cup \{\{x \in X : x > a\} : a \in X\}$.

Definition. Let X be a LOTS and $a \in X$. We say that a is *left-isolated* (resp. *right-isolated*) if $(< a) = \emptyset$ (resp. $(> a) = \emptyset$) or there exists $x \in X$ with $x < a$ (resp. $x > a$) such that $]x, a[= \emptyset$ (resp. $]a, x[= \emptyset$). In addition, a point a is *isolated* when it is left and right-isolated. The sets of left-isolated, right-isolated and isolated points are denoted by $\text{Ais}(X)^-$, $\text{Ais}(X)^+$ and $\text{Ais}(X)$, respectively.

Given a second-countable LOTS X , one may construct a fractal structure Γ_{ord} provided that the sets of left and right-isolated points are countable and there exists a dense countable subset D' . Denote by $D = D' \cup \text{Ais}(X)^- \cup \text{Ais}(X)^+$. The procedure is the following:

1. Choose the first element x_1 in D and define the first level Γ_1 in the next way:
 - $\Gamma_1 = \{(\leq x_1^-), \{x_1\}, (\geq x_1^+)\}$ if $x_1 \in \text{Ais}(X)$.
 - $\Gamma_1 = \{(\leq x_1^-), (\geq x_1)\}$ if $x_1 \in \text{Ais}(X)^- \setminus \text{Ais}(X)^+$.
 - $\Gamma_1 = \{(\leq x_1), (\geq x_1^+)\}$ if $x_1 \in \text{Ais}(X)^+ \setminus \text{Ais}(X)^-$.
 - $\Gamma_1 = \{(\leq x_1), (\geq x_1)\}$ if $x_1 \in \text{Ais}(X)^- \cup \text{Ais}(X)^+$.
2. To define the following levels, we select the first element in D contained in the element of the previous level and split it into similar intervals as before.

This construction gives a tiling finite starbase fractal structure. If μ is a non-atomic measure ($\mu(\{x\}) = 0, \forall x \in X$) on the Borel σ -algebra of X , this fractal structure is μ -disjoint and adding the condition to X of being compact makes it Cantor-complete. The next figure shows a representation of the fractal structure Γ_{ord} , where each colorbar represents the point used to split up the space in each iteration:



Fundamental Theorem of Calculus

Let X be a compact second-countable LOTS and μ a non-atomic measure on the Borel σ -algebra. Consider the previous fractal structure Γ_{ord} .

Definition. A function $f : X \rightarrow \mathbb{R}$ is μ -differentiable at $z \in X \setminus \text{Ais}(X)$ if there exists $L \in \mathbb{R}$, denoted by $L = f'(z)$, such that for all $\varepsilon > 0$, there is an interval $]a, b[$ containing z verifying

$$|f(x) - f(z) - L\mu'[z, x]| \leq \varepsilon |\mu'[z, x]|, \text{ where } \mu'[z, x] = \begin{cases} \mu[z, x] & \text{if } z \leq x \\ -\mu[x, z] & \text{if } z > x \end{cases}.$$

Theorem. Let $F : X \rightarrow \mathbb{R}$ be a μ -differentiable function at every $z \in X \setminus \text{Ais}(X)$ and write $X = [a, b]$ and $\text{Ais}(X)^- \setminus \{a\} = \{x_k : k \in \mathbb{N}\}$. Suppose that $a, b \notin \text{Ais}(X)$, $0 < \mu(X) < \infty$ and $\sum_{k=1}^{\infty} F(x_k^-) - F(x_k)$ is convergent. Then $F' \in \mathcal{HK}(X, \mu, \Gamma_{\text{ord}})$ with

$$(\Gamma) \int_{[a,b]} F' d\mu = F(b) - F(a) + \sum_{k=1}^{\infty} F(x_k^-) - F(x_k).$$

Conclusions

- The HK-integral with fractal structures verifies a similar result to *Cauchy's Criterion* and adding the tiling condition to the fractal structure ensures that *additivity* and *Henstock's Lemma* still hold.
- Also, *Monotone* and *Dominated Convergence theorems* may be obtained.
- A version of the *Fundamental Theorem of Calculus* is given in the context of LOTS.

References

- [1] J.D. DEPREE, C.W. SWARTZ, *Introduction to Real Analysis*, John Wiley & Sons, 1988.
- [2] J.F. GÁLVEZ-RODRÍGUEZ, M.A. SÁNCHEZ-GRANERO, *Constructing a Linearly Ordered Topological Space from a Fractal Structure: A Probabilistic Approach*, Mathematics. **10**(23), (2022), 4518.