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EXISTENCE OF POSITIVE SOLUTIONS TO A SEMIPOSITONE SINGULAR ψ-RIEMANN-LIOUVILLE FRACTIONAL BOUNDARY VALUE PROBLEM

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Introduction

We consider the fractional differential equation

$$D_{a+}^{\alpha,\psi}u(t) + \lambda f(t, u(t)) = 0, \quad t \in (a,b), \tag{1}$$

subject to the nonlocal boundary conditions

$$u^{(i)}(a) = 0, \quad i = 0, 1, \dots, n-2; \quad D_{a+}^{\varsigma,\psi}u(b) = \sum_{i=1}^{m} \int_a^b D_{a+}^{\varrho_i,\psi}u(s)\, d\mathcal{H}_i(s), \tag{2}$$

where $0 \le a < b$, $\alpha > 0$, $\alpha \in (n-1, n]$, $n \in \mathbb{N}$, $n \ge 3$, $\psi \in C^n[a,b]$ with $\psi'(t) > 0$ for all $t \in [a,b]$, $D_{a+}^{k,\psi}u$ denotes the ψ-Riemann-Liouville fractional derivative of function u of order k, for $k \in \{\alpha, \varsigma, \varrho_i, \ i = 1, \dots, m\}$, $m \in \mathbb{N}$, $1 \le \varsigma < \alpha - 1$, $0 \le \varrho_i \le \varsigma$, $i = 1, \dots, m$, λ is a positive parameter, the function f may change sign and may be singular at the points $t = a$ and/or $t = b$, and $\mathcal{H}_i : [a,b] \to \mathbb{R}$, $i = 1, \dots, m$ are bounded variation functions.

We give intervals for the parameter λ such that the problem (1)-(2) has at least one positive solution. Since f can have negative values, then our problem is called a semipositone fractional boundary value problem. In the proofs of the main results we apply the Guo-Krasnosel'skii fixed point theorem. The ψ-Riemann-Liouville fractional derivative generalizes the Riemann-Liouville derivative (for $\psi(t) = t$), and the Hadamard derivative (for $\psi(t) = \ln t$).

Auxiliary results

We consider the fractional differential equation

$$D_{a+}^{\alpha,\psi}u(t) + k(t) = 0, \quad t \in (a,b), \tag{3}$$

with the boundary conditions (2), where $k \in C(a,b) \cap L^1(a,b)$. We denote by

$$\Delta = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\varsigma)}(\psi(b)-\psi(a))^{\alpha-\varsigma-1} - \sum_{i=1}^{m} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\varrho_i)} \int_a^b (\psi(s)-\psi(a))^{\alpha-\varrho_i-1}\, d\mathcal{H}_i(s).$$

Lemma 1 *If $\Delta \ne 0$, then the solution of problem (3)-(2) is*

$$u(t) = \int_a^b \mathcal{G}(t,s)k(s)\, ds, \quad t \in [a,b], \tag{4}$$

where the Green function $\mathcal{G}$ is given by

$$\mathcal{G}(t,s) = g_1(t,s) + \frac{(\psi(t)-\psi(a))^{\alpha-1}}{\Delta} \sum_{i=1}^{m} \left(\int_a^b g_{2i}(\tau,s)\, d\mathcal{H}_i(\tau) \right), \quad t, s \in [a,b],$$

$$g_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \psi'(s)\left[(\psi(t)-\psi(a))^{\alpha-1} \left(\frac{\psi(b)-\psi(s)}{\psi(b)-\psi(a)} \right)^{\alpha-\varsigma-1} - (\psi(t)-\psi(s))^{\alpha-1} \right], & a \le s \le t \le b, \\ \psi'(s)(\psi(t)-\psi(a))^{\alpha-1} \left(\frac{\psi(b)-\psi(s)}{\psi(b)-\psi(a)} \right)^{\alpha-\varsigma-1}, & a \le t \le s \le b, \end{cases}$$

$$g_{2i}(\tau,s) = \frac{1}{\Gamma(\alpha-\varrho_i)} \begin{cases} \psi'(s)\left[(\psi(\tau)-\psi(a))^{\alpha-\varrho_i-1} \left(\frac{\psi(b)-\psi(s)}{\psi(b)-\psi(a)} \right)^{\alpha-\varsigma-1} - (\psi(\tau)-\psi(s))^{\alpha-\varrho_i-1} \right], \\ \qquad a \le s \le \tau \le b, \\ \psi'(s)(\psi(\tau)-\psi(a))^{\alpha-\varrho_i-1} \left(\frac{\psi(b)-\psi(s)}{\psi(b)-\psi(a)} \right)^{\alpha-\varsigma-1}, \quad a \le \tau \le s \le b, \end{cases}$$

for $i = 1, \dots, m$.

Lemma 2 *Assume that $\Delta > 0$ and $\mathcal{H}_i : [a,b] \to \mathbb{R}$, $i = 1, \dots, m$ are nondecreasing functions. Then the function $\mathcal{G}$ is a continuous function on $[a,b] \times [a,b]$ and satisfies the conditions:*

a) $\mathcal{G}(t,s) \le \mathcal{J}(s)$, for all $t, s \in [a,b]$, where

$$\mathcal{J}(s) = h_1(s) + \frac{(\psi(b)-\psi(a))^{\alpha-1}}{\Delta} \sum_{i=1}^{m} \left(\int_a^b g_{2i}(\tau,s)\, d\mathcal{H}_i(\tau) \right), \quad s \in [a,b],$$

$$h_1(s) = \frac{1}{\Gamma(\alpha)} \psi'(s)(\psi(b)-\psi(s))^{\alpha-\varsigma-1}[(\psi(b)-\psi(a))^{\varsigma} - (\psi(b)-\psi(s))^{\varsigma}], \quad s \in [a,b].$$

b) $\mathcal{G}(t,s) \ge (\gamma(t))^{\alpha-1}\mathcal{J}(s)$, for all $t, s \in [a,b]$, where $\gamma(t) = \frac{\psi(t)-\psi(a)}{\psi(b)-\psi(a)}$, $t \in [a,b]$.

c) $\mathcal{G}(t,s) \le \theta(\gamma(t))^{\alpha-1}$, for all $t, s \in [a,b]$, where

$$\theta = C_0(\psi(b)-\psi(a))^{\alpha-1} \left[\frac{1}{\Gamma(\alpha)} + \frac{1}{\Delta} \sum_{i=1}^{m} \left(\int_a^b \frac{1}{\Gamma(\alpha-\varrho_i)} (\psi(\tau)-\psi(a))^{\alpha-\varrho_i-1}\, d\mathcal{H}_i(\tau) \right) \right] > 0,$$

with $C_0 = \sup_{s \in [a,b]} \psi'(s)$.

Lemma 3. *Assume that $\Delta > 0$, $\mathcal{H}_i : [a,b] \to \mathbb{R}$, $i = 1, \dots, m$ are nondecreasing functions, and $k(t) \ge 0$ for all $t \in (a,b)$. Then the solution u of problem (3)-(2) given by (4) satisfies the inequality $u(t) \ge (\gamma(t))^{\alpha-1}u(\zeta)$ for all $t, \zeta \in [a,b]$.*

Main results

We present below the assumptions that we will use in the sequel.

$(A1)$ $\alpha > 0$, $\alpha \in (n-1, n]$, $n \in \mathbb{N}$, $n \ge 3$, $\psi \in C^n([a,b])$ with $\psi'(t) > 0$ for all $t \in [a,b]$, $1 \le \varsigma < \alpha - 1$, $0 \le \varrho_i \le \varsigma$ for all $i = 1, \dots, m$, $\lambda > 0$, $\mathcal{H}_i : [a,b] \to \mathbb{R}$, $i = 1, \dots, m$ are nondecreasing functions, and $\Delta > 0$.

$(A2)$ The function $f \in C((a,b) \times \mathbb{R}_+, \mathbb{R})$ may be singular at $t = a$ and/or $t = b$, and there exist the functions $p, q \in C((a,b), \mathbb{R}_+)$, $w \in C([a,b] \times \mathbb{R}_+, \mathbb{R}_+)$ such that

$$-q(t) \le f(t,y) \le p(t)w(t,y), \quad \forall\, t \in (a,b), \ y \in \mathbb{R}_+,$$

with $0 < \int_a^b q(t)\, dt < \infty$ and $0 < \int_a^b p(t)\, dt < \infty$, ($\mathbb{R}_+ = [0, \infty)$).

$(A3)$ There exist c, $d \in (a,b)$, $c < d$ such that $f_\infty = \lim_{y \to \infty,\, t \in [c,d]} f(t,y)/y = \infty$.

$(A4)$ There exist $\widetilde{c}$, $\widetilde{d} \in (a,b)$, $\widetilde{c} < \widetilde{d}$ such that $\liminf_{y \to \infty} \min_{t \in [\widetilde{c}, \widetilde{d}]} f(t,y) > \Lambda_0$, with $\Lambda_0 = \left(2\theta \int_a^b q(s)\, ds \right) \times \left((\gamma(\widetilde{c}))^{\alpha-1} \int_{\widetilde{c}}^{\widetilde{d}} \mathcal{J}(s)\, ds \right)^{-1}$, and $w_\infty = \lim_{y \to \infty} \max_{t \in [a,b]} w(t,y)/y = 0$, where $\mathcal{J}$ and θ are given in Lemma 2.

We consider the fractional differential equation

$$D_{a+}^{\alpha,\psi}v(t) + \lambda(f(t, [v(t) - \lambda z(t)]^*) + q(t)) = 0, \quad t \in (a,b), \tag{5}$$

with the boundary conditions

$$v^{(i)}(a) = 0, \quad i = 0, 1, \dots, n-2; \quad D_{a+}^{\varsigma,\psi}v(b) = \sum_{i=1}^{m} \int_a^b D_{a+}^{\varrho_i,\psi}v(s)\, d\mathcal{H}_i(s), \tag{6}$$

where $\vartheta(t)^* = \vartheta(t)$ if $\vartheta(t) \ge 0$, and $\vartheta(t)^* = 0$ if $\vartheta(t) < 0$.

Here $z(t) = \int_a^b \mathcal{G}(t,s)q(s)\, ds$, $t \in [a,b]$ is the solution of the problem

$$\begin{cases} D_{a+}^{\alpha,\psi}z(t) + q(t) = 0, \quad t \in (a,b), \\ z^{(i)}(a) = 0, \quad i = 0, 1, \dots, n-2; \quad D_{a+}^{\varsigma,\psi}z(b) = \sum_{i=1}^{m} \int_a^b D_{a+}^{\varrho_i,\psi}z(s)\, d\mathcal{H}_i(s). \end{cases}$$

Under assumptions $(A1)$, $(A2)$, we have $z(t) \ge 0$ for all $t \in [a,b]$. We show that there exists a solution v of problem (5)-(6), with $v(t) \ge \lambda z(t)$ on $[a,b]$, and $v(t) > \lambda z(t)$ on (a,b). In this case $u = v - \lambda z$ represents a positive solution of problem (1)-(2). Hence next we study the problem (5)-(6).

By using Lemma 1, v is a solution of problem (5)-(6) if and only if v is a solution of equation

$$v(t) = \lambda \int_a^b \mathcal{G}(t,s)(f(s, [v(s) - \lambda z(s)]^*) + q(s))\, ds, \quad t \in [a,b]. \tag{7}$$

We consider the Banach space $\mathcal{X} = C[a,b]$ with the supremum norm $\|v\| = \sup_{t \in [a,b]} |v(t)|$, and we define the cone $\mathcal{P} = \left\{ v \in \mathcal{X}, \ v(t) \ge (\gamma(t))^{\alpha-1}\|v\|, \ \forall\, t \in [a,b] \right\}$.

For $\lambda > 0$ we introduce the operator

$$\mathcal{L}v(t) = \lambda \int_a^b \mathcal{G}(t,s)(f(s, [v(s) - \lambda z(s)]^*) + q(s))\, ds,$$

for $t \in [a,b]$, and $v \in \mathcal{X}$. It is clear that v is a solution of equation (7) (or equivalently of problem (5)-(6)) if and only if v is a fixed point of operator $\mathcal{L}$.

Lemma 4. *If $(A1)$ and $(A2)$ hold, then the operator $\mathcal{L} : \mathcal{P} \to \mathcal{P}$ is a completely continuous operator.*

By applying the Guo-Krasnosel'skii fixed point theorem of cone expansion and compression of norm type for operator $\mathcal{L}$, we obtain the following main results for our problem (1)-(2).

Theorem 1. *Assume that $(A1)$, $(A2)$ and $(A3)$ hold. Then there exits $\lambda_1 > 0$ such that for all $\lambda \in (0, \lambda_1]$, the boundary value problem (1)-(2) has at least one positive solution $u(t)$, $t \in [a,b]$.*

Theorem 2. *Suppose that $(A1)$, $(A2)$ and $(A4)$ hold. Then there exists $\lambda_2 > 0$ such that for any $\lambda \ge \lambda_2$, the boundary value problem (1)-(2) has at least one positive solution $u(t)$, $t \in [a,b]$.*

Theorem 3. *Assume that $(A1)$, $(A2)$ and*

$(\widetilde{A4})$ *There exist $\widetilde{c}$, $\widetilde{d} \in (a,b)$, $\widetilde{c} < \widetilde{d}$ such that $\widetilde{f}_\infty = \lim_{y \to \infty} \min_{t \in [\widetilde{c}, \widetilde{d}]} f(t,y) = \infty$ and $w_\infty = \lim_{y \to \infty} \max_{t \in [a,b]} w(t,y)/y = 0$,*

hold. Then there exists $\widetilde{\lambda}_2 > 0$ such that for any $\lambda \ge \widetilde{\lambda}_2$, the boundary value problem (1)-(2) has at least one positive solution $u(t)$, $t \in [a,b]$.

References

1. A. Tudorache, R. Luca, Positive solutions for a semipositone singular ψ-Riemann-Liouville fractional boundary value problem, *Mathematics*, **13** (20) (3292), (2025), 1-24.