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On the Stability of Solutions to Certain Tribonacci-Type Difference Equation Systems

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Abstract

In this paper, we study a nonlinear system of rational difference equations of order (p+1). We obtain explicit formulas for its solutions and investigate their stability and asymptotic behavior. Under suitable conditions on the initial values, several qualitative properties of the solutions are established, illustrating the long-term dynamics of the system.
Keywords: Tribonacci numbers, Stability, Equilibrium point, Solutions, System of Difference equations.

INTRODUCTION

The theory of difference equations developed greatly during the last twenty-five years of the twentieth century. The applications of the theory of difference equations is rapidly increasing to various fields such as numerical analysis, biology, economics, control theory, finite mathematics and computer science. Thus, there is every reason for studying the theory of difference equations as a well deserved discipline.
In 2015 [2], Halim Yacine studied the stability and the periodicity for the solutions of the following system of difference equations

$$\begin{cases} x_{n+1} = \frac{1}{1+y_n}, \\ y_{n+1} = \frac{1}{1+x_n} \end{cases} \quad n \in \mathbb{N}_0 \tag{0.1}$$

In 2021 [4], I. Talha, S. Badidja investigated the periodicity of the solutions of the following system of rational difference equations:

$$\begin{cases} x_{n+1} = \frac{y_n(x_{n-(p-1)}+y_{n-p})}{y_{n-p}+x_{n-(p-1)}-y_n}, \\ y_{n+1} = \frac{x_{n-(p-2)}(x_{n-(p-2)}+y_{n-(p-1)})}{2x_{n-(p-2)}+y_{n-(p-1)}}, \end{cases} \tag{0.2}$$

PRELIMINARIES

Definition 1 (Tribonacci sequence) *[3] The Tribonacci sequence $\{T_n\}_{n\geq 0}$ is defined by the following thrid-order linear recurrent relation*

$$\begin{cases} T_n = T_{n-1} + T_{n-2} + T_{n-3}, n \geq 4 \\ T_1 = T_2 = 1 \\ T_3 = 2 \end{cases} \tag{0.3}$$

where the first terms of the Tribonacci sequence are given by: $1, 1, 2, 3, 5, 8, 13, \ldots$

Definition 2 (Stability) *[1] Let $\overline{W}$ be an equilibrium point and $\|.\|$ be any norm (e.g., the Euclidean norm).*

1. *The equilibrium point $\overline{W}$ is called stable (or locally stable) if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\|W_0 - \overline{W}\| < \delta$ implies $\|W_n - \overline{W}\| < \epsilon$ for $n \geq 0$.*
2. *The equilibrium point $\overline{W}$ is called asymptotically stable (or locally asymptotically stable) if it is stable and there exists $\gamma > 0$ such that $\|W_0 - \overline{W}\| < \gamma$ implies*
$$\|W_n - \overline{W}\| \longrightarrow 0, n \longrightarrow +\infty.$$
3. *The equilibrium point $\overline{W}$ is said to be global attractor (respectively global attractor with basin of attraction a set $G \subseteq I^k \times J^k$), if for every W_0 (respectively for every $W_0 \in G$)*
$$\|W_n - \overline{W}\| \longrightarrow 0, n \longrightarrow +\infty.$$
4. *The equilibrium point $\overline{W}$ is called globally asymptotically stable (respectively globally asymptotically stable relative to G) if it is asymptotically stable, and if for every W_0 (respectively for every $W_0 \in G$),*
$$\|W_n - \overline{W}\| \longrightarrow 0, n \longrightarrow +\infty.$$
5. *The equilibrium point $\overline{W}$ is called unstable if it is not stable.*

Theorem 1 *[1] If all the eigenvalues of the Jacobian matrix A lie in the open unit disk $|\lambda| < 1$, then the equilibrium point $\overline{W}$ is asymptotically stable.*
On the other hand, if at least one eigenvalue of the Jacobian matrix A have absolute value greater than one, then the equilibrium point $\overline{W}$ is unstable.

METHOD

In this work we interested the form of solutions, stability character and asymptotic of the system non-linear of difference equations in order $p+1$.

$$\begin{cases} x_{n+1} = \frac{1}{y_{n-(p-1)}(x_{n-p}\pm 1)+1}, \\ y_{n+1} = \frac{1}{x_{n-(p-1)}(y_{n-p}\pm 1)+1} \end{cases} \quad n \in \mathbb{N}_0, \quad p \geq 1 \tag{0.4}$$

where $x_{-p}, x_{-(p-1)}, \ldots, x_0, y_{-p}, y_{-(p-1)}, \ldots, y_0$ are real initial values with the certain conditions.

Existence of equilibrium point

We have the following system of equations

$$\begin{cases} \bar{x} = \frac{1}{\bar{y}(\bar{x}+1)+1} \\ \bar{y} = \frac{1}{\bar{x}(\bar{y}+1)+1} \end{cases} \tag{0.5}$$

In (0.5), by subtracting the second equation from the first equation and by some computations, we get

$$\bar{x} - \bar{y} = \frac{1}{1+\bar{x}\bar{y}+\bar{y}} - \frac{1}{1+\bar{y}\bar{x}+\bar{x}}$$

and by some operations we get the result

$$(\bar{x}-\bar{y})[(1+\bar{x}\bar{y}+\bar{y})(1+\bar{y}\bar{x}+\bar{x})-1] = 0.$$

Hence,

$$\bar{x} = \bar{y}$$

Then, the system (0.5) can be written as

$$\begin{cases} \bar{x}^3 + \bar{x}^2 + \bar{x} - 1 = 0 \\ \bar{y}^3 + \bar{y}^2 + \bar{y} - 1 = 0 \end{cases} \tag{0.6}$$

and the characteristic equation

$$\bar{x}^3 + \bar{x}^2 + \bar{x} - 1 = 0 \tag{0.7}$$

is having three roots a, b and c where,

$$\begin{cases} a = \frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3} \\ b = \frac{1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega^2\sqrt[3]{19-3\sqrt{33}}}{3} \\ c = \frac{1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega\sqrt[3]{19-3\sqrt{33}}}{3} \end{cases}$$

and $\omega = \frac{-1+i\sqrt{3}}{2}$ is a primitive cube root of unity. Hence the unique real positive equilibrium point of system (0.4) is given by

$$W = (a, a, \ldots, a, a, a, \ldots, a) \in I^{p+1} \times J^{p+1}$$

Which a is the real root of the characteristic equation: $x^3 + x^2 + x - 1$.

Theorem 2 *The equilibrium point of system (0.4) is locally asymptotically stable.*

Proof
Let $I = J = (0, +\infty)$ and consider the functions:

$$f : I^{p+1} \times J^{p+1} \longrightarrow I, \quad g : I^{p+1} \times J^{p+1} \longrightarrow J$$

defined by

$$f(x_n, \ldots, x_{n-p}, y_n, \ldots, y_{n-p}) = \frac{1}{y_{n-(p-1)}(x_{n-p}+1)+1}.$$
$$g(x_n, \ldots, x_{n-p}, y_n, \ldots, y_{n-p}) = \frac{1}{x_{n-(p-1)}(y_{n-p}+1)+1}$$

and we have the following transformation

$$(x_n, \ldots, x_{n-(p-1)}, y_n, \ldots, y_{n-(p-1)}) = (f_1, \ldots, f_p, g_1, \ldots, g_p)$$

with

$$\begin{cases} f(x_n, x_{n-1}, \ldots, x_{n-p}, y_n, y_{n-1}, \ldots, y_{n-p}) = \frac{1}{y_{n-(p-1)}(x_{n-p}+1)+1} \\ f_1(x_n, x_{n-1}, \ldots, x_{n-p}, y_n, y_{n-1}, \ldots, y_{n-p}) = x_n \\ \ldots\ldots\ldots \\ f_p(x_n, x_{n-1}, \ldots, x_{n-p}, y_n, y_{n-1}, \ldots, y_{n-p}) = x_{n-(p-1)} \\ g(x_n, x_{n-1}, \ldots, x_{n-p}, y_n, y_{n-1}, \ldots, y_{n-p}) = \frac{1}{x_{n-(p-1)}(y_{n-p}+1)+1} \\ g_1(x_n, x_{n-1}, \ldots, x_{n-p}, y_n, y_{n-1}, \ldots, y_{n-p}) = y_n \\ \ldots\ldots\ldots \\ g_p(x_n, x_{n-1}, \ldots, x_{n-p}, y_n, y_{n-1}, \ldots, y_{n-p}) = y_{n-(p-1)}. \end{cases}$$

The linearised system associated to the non-linear system (0.4) about the positive equilibrium point $W = (a, a, \ldots, a, a, a, \ldots, a) \in I^{p+1} \times J^{p+1}$ is given by

$$W_{n+1} = MW_n, \tag{0.8}$$

where M is $2p \times 2p$ Jacobian matrix.
Hence, the characteristic equation of the Jacobian matrix M is given as

$$(\lambda^2 + (a-1)\lambda + a^3)(\lambda^2 - (a-1)\lambda + a^3).$$

Numerically we get

$$|\lambda_1| = |\lambda_2| = |\lambda_3| = \ldots = |\lambda_{2p}| \simeq 0.40089 < 1$$

Therfore, the equilibrium point W is locally asymptotically stable.

Theorem 3 *The equilibrium point of system $(0.4)$is globally asymptotically stable.*

Proof
Let $\{x_n, y_n\}_{n\geq 0}$ be a solution of system (0.4).
By definition 2 we just need only to prove that W is global attractor, that is

$$\lim_{n\to+\infty} (x_n, y_n) = W.$$

By evaluating the limit of the solutions, we obtain

$$\lim_{n\longrightarrow+\infty} y_{2pn-i} = a, \quad \lim_{n\longrightarrow+\infty} y_{2pn-(i-1)} = a$$

So

$$\lim_{n\to+\infty} (x_n, y_n) = W.$$

Hence, the equilibrium point W is globally asymptotically stable.

NUMERICAL EXAMPLE/ RESULTS

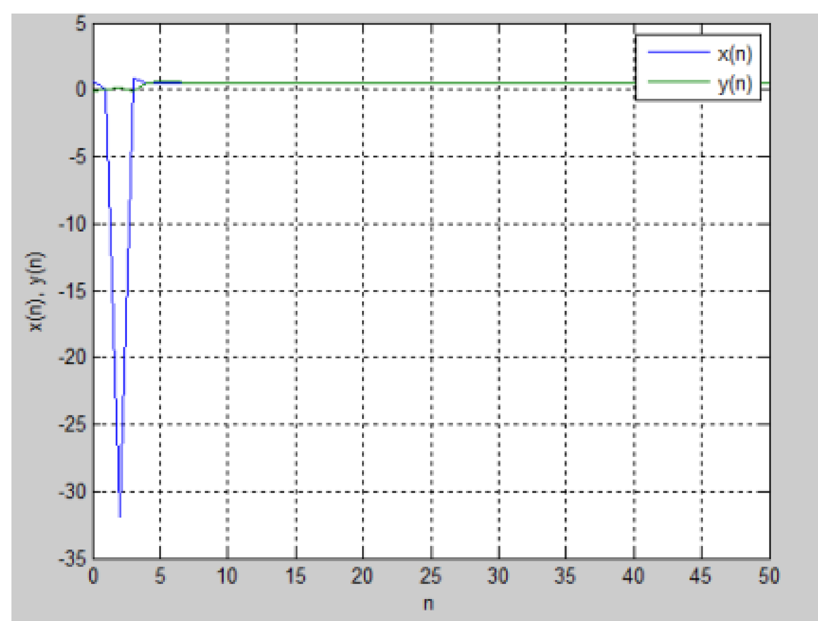

Figure 1: The sequences $(x_n)_{n\geq 0}$ (blue) and $(y_n)_{n\geq 0}$ (green) of solution of the system (0.4)

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