

Study solutions to the coupled nonlinear fractional DSW model

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INTRODUCTION & AIM

Regarding the Drinfel'd-Sokolov-Wilson (DSW) equation, modified extended direct algebraic methods yielded solutions in the form of bell, anti-bell, periodic, and dark solitary waves in 2017 [1], while series solutions were obtained using the Adomian decomposition method in 2022 [2].

In the field of fractional calculus, many researchers are focused on constructing exact solutions for fractional differential equations. In 2023, singular bright, dark, periodic, bell, and lump-type water wave solutions to the coupled nonlinear fractional FDSW) model with the Beta derivative (see **Eq. (1)**) were explored using the generalized rational exponential function method [3].

$$\begin{cases} D_t^\alpha \Phi + a\Psi D_x \Psi & = 0 \\ D_t^\alpha \Psi + \gamma_1 \Psi D_x \Phi + \lambda_1 \Phi D_x \Psi + \eta_1 D_{xxx} \Psi & = 0. \end{cases} \quad (1)$$

However, the exact form of Weierstrass function-type solutions has not yet been established. Therefore, in this research, we aim to construct new travelling wave solutions for the FDSW model using the extended complex method and compare our results to those that are already known. Taking the following travelling wave transformation on **Eq. (1)**, we have

$$\begin{cases} \Phi(x, t) = \Phi(z), \Psi(x, t) = \Psi(z), z = \kappa_1 \left(x - \frac{\omega_1}{\alpha} \left(t + \frac{1}{\Gamma(\alpha)} \right)^\alpha \right) \\ \begin{cases} a\kappa_1 \Psi \Psi' - \kappa_1 \omega_1 \Phi' & = 0 \\ -\kappa_1 \omega_1 \Psi' + \gamma_1 \kappa_1 \Psi \Phi' + \lambda_1 \kappa_1 \Phi \Psi' + \eta_1 \kappa_1^3 \Psi''' & = 0. \end{cases} \end{cases} \quad (2)$$

METHOD [4]

Step 1 Using the transform $T : u(x, t) = u(z), w(x, t) = w(z), (x, t) \rightarrow z$ to reduce a given system of partial differential equations to the following system of ordinary differential equations

$$\begin{cases} P(u, u', \dots, u^{(m)}, w, w', \dots, w^{(s)}) = 0 \\ Q(u, u', \dots, u^{(m)}, w, w', \dots, w^{(s)}) = 0. \end{cases} \quad (3)$$

Then obtain a ordinary differential equation

$$P(w, w', \dots, w^{(m)}) = bw^n + c. \quad (4)$$

Step 2 Determine whether all meromorphic solutions $u(z)$ and $w(z)$ of Eq. (4) belong to the class **W** by using Nevanlinna's value distribution theory and Painlevé test.

Step 3 Substituting $w(z) = \sum_{k=-q}^{\infty} c_k z^k$ into (4) to determine that the (weak) $\langle p, q \rangle$ condition holds by using the Painlevé test.

Step 4 Building the non-constant polynomial and transcendental entire function solutions to Eq. (3) or (4) using their forms to be determined. By indeterminate relations (see Huang et al.(2014)), building the elliptic, rational and simply periodic solutions $u(z), w(z)$ of Eq. (3) or (4) with pole at $z = 0$, respectively.

Step 5 Taking the inverse transform T^{-1} .

RESULTS & DISCUSSION

Eq. (2) can be reduce into Eq. (5)

$$A\Psi'' + B\Psi + C\Psi^3 = 0 \quad \text{and} \quad \Phi = \frac{a\Psi^2}{2\omega_1} \quad (5)$$

where $A = 6\eta_1\kappa_1^2\omega_1$, $B = -6\omega_1^2$, $C = a(\lambda_1 + 2\gamma_1)$. (6)

Applying the method and some computation, we obtain the following solution to Eq. (5)

I. Rational solution: B must zero, $\Psi(z) = \pm \frac{\sqrt{-\frac{2A}{C}}}{z - z_0}$.

II. Elliptic solutions:

$$c_1 = \sqrt{-\frac{2A}{C}} \quad \Psi(z) = \frac{\sqrt{-\frac{2A}{C}}}{2} \frac{\wp'(z, -\frac{54A^3g_3 - B^3}{9A^2B}, g_3)}{\wp(z, -\frac{54A^3g_3 - B^3}{9A^2B}, g_3) - \frac{B}{6A}}$$

$$c_1 = -\sqrt{-\frac{2A}{C}} \quad \Psi(z) = \frac{-\sqrt{-\frac{2A}{C}}}{2} \frac{\wp'(z, -\frac{54A^3g_3 - B^3}{9A^2B}, g_3)}{\wp(z, -\frac{54A^3g_3 - B^3}{9A^2B}, g_3) - \frac{B}{6A}}$$

III. Trigonometric solutions

$$\begin{aligned} \Psi(z) &= \pm \frac{\sqrt{2} \sqrt{-\frac{A}{C}} \sqrt{\frac{B}{A}}}{2 \cos\left(\frac{\sqrt{\frac{B}{A}}z}{2}\right) \sin\left(\frac{\sqrt{\frac{B}{A}}z}{2}\right)} \\ \Psi(z) &= \pm \sqrt{-\frac{A}{C}} \sqrt{-\frac{B}{A}} \cot\left(\frac{\sqrt{2} \sqrt{-\frac{B}{A}}z}{2}\right) \end{aligned}$$

IV. Exponential solutions

$$\begin{aligned} \Psi(z) &= \pm \frac{\sqrt{AB} \sqrt{-AC} \left(e^{\frac{\sqrt{2} \sqrt{AB}z}{A}} - \xi \right)}{AC \left(e^{\frac{\sqrt{2} \sqrt{AB}z}{A}} + \xi \right)} \\ \Psi(z) &= \pm \frac{\sqrt{AB} \sqrt{-AC} \left(e^{-\frac{\sqrt{2} \sqrt{AB}z}{A}} - \xi \right)}{AC \left(e^{-\frac{\sqrt{2} \sqrt{AB}z}{A}} + \xi \right)}. \end{aligned}$$

CONCLUSION

Taking the inverse travelling wave transformation on the above solutions, the exact solutions of Eq. (1) will be obtain directly. Comparing Ref. [3], the above elliptic solutions and the exponential solutions with $\xi \neq 1$ are seem to be new.

REFERENCES

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