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On the Spectral Theory of Regularized Quasi-Semigroups

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INTRODUCTION & AIM

Historical background. In 1953, Tosio Kato studied the time-dependent evolution equation in a Banach space X:

$$x'(t) = A(t)\,x(t) + f(t), \quad t \geq 0, \quad x(0) = x_0,$$

where $A(t)$ is a closed operator with time-independent, dense domain $\mathcal{D}$. The solution is expressed by the **evolution operator** $U(t,s)$ $(s \leq t)$ via

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s)\,ds.$$

$U(t,s)$ satisfies: $U(r,r) = I,\ U(t,r)U(r,s) = U(t,s),$ strong continuity, and $\partial_t U(t,r)x$ exists and is continuous.

From evolution operators to quasi-semigroups. Setting $R(t,s) := U(t+s,t)$ turns the evolution-operator family into a C_0**-quasi-semigroup**, a two-parameter semigroup theory introduced by Leiva & Bárcenas (1991). They established basic properties and gave sufficient conditions for well-posedness of the associated Cauchy problem.

C**-regularization.** When initial data must be regularized by an injective operator $C \in \mathcal{B}(X)$, the resulting family $\{K(t,s)\}_{t,s\geq 0}$ (Janfada, 2010) is called a C**-quasi-semigroup**. It generalizes C_0-quasi-semigroups $(C = I)$ and allows treating ill-posed problems inaccessible to the classical theory.

Aim. Establish **spectral inclusion theorems** relating, for $t \geq s \geq 0$:

$$e^{s\,\sigma_\star(A(t))} \ \subseteq\ \sigma_\star(C,\, K(t-s,s)) \setminus \{0\}$$

for the spectra $\sigma_\star \in$ {ordinary, point, approximate point, essential, residual, ascent, descent, Saphar, Kato, quasi-Fredholm}. These inclusions give *qualitative information on solutions* directly from spectral properties of the generator family $\{A(t)\}_{t\geq 0}$.

Notation. Let X be a Banach space, $\mathcal{B}(X)$ bounded operators, $\mathcal{C}(X)$ closed operators. For $T \in \mathcal{C}(X)$:
- $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T$ not bijective$\}$
- $\sigma_p(T) = \{\lambda : \ker(\lambda I - T) \neq \{0\}\}$
- $\sigma_{ap}(T) = \{\lambda : \lambda I - T$ not bounded below$\}$
- $\sigma_r(T) = \{\lambda : \overline{\mathrm{Rg}(\lambda I - T)} \neq X\}$
- $\sigma_e(T) = \{\lambda : \lambda I - T \notin \Phi(X)\}$ (essential spectrum)

PRELIMINARIES

C_0**-Semigroup** $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$: (i) $T(0) = I$; (ii) $T(t+s) = T(t)T(s)$; (iii) $\lim_{t\to 0^+} T(t)x = x$.
Generator: $Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t},\ D(A) = \{x :$ limit exists in $X\}$.

C**-Semigroup** $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ [Davies & Pang, 1987]: (i) $S(0) = C$; (ii) $S(t)S(s) = CS(t+s)$; (iii) $t \mapsto S(t)x$ continuous.
Generator: $Ax = C^{-1}\lim_{t\to 0^+} \frac{S(t)x - Cx}{t},\ D(A) = \left\{x : \lim \frac{S(t)x - Cx}{t} \in \mathrm{Rg}(C)\right\}$.

C_0**-Quasi-Semigroup** [Leiva & Bárcenas, 1991]. $\{R(t,s)\}_{t,s\geq 0} \subset \mathcal{B}(X)$:
- (i) $R(t,0) = I$;
- (ii) $R(t,s+r) = R(t+r,s)R(t,r)$;
- (iii) $(t,s) \mapsto R(t,s)x$ continuous;
- (iv) $\|R(t,s)\| \leq M(s)$, M increasing.

Generator: $A(t)x = \lim_{s\to 0^+} \frac{R(t,s)x - x}{s},\ x \in \mathcal{D}$.

C**-Quasi-Semigroup** [Janfada, 2010]. Let $C \in \mathcal{B}(X)$ injective. $\{K(t,s)\}_{t,s\geq 0} \subset \mathcal{B}(X)$:
- (i) $K(t,0) = C$;
- (ii) $CK(t,s+r) = K(t+r,s)K(t,r)$;
- (iii) $(t,s) \mapsto K(t,s)x$ continuous;
- (iv) $\|K(t,s)\| \leq M(t+s)$, M increasing.

Generator: $A(t)x = C^{-1}\lim_{s\to 0^+} \frac{K(t,s)x - Cx}{s},\ x \in \mathcal{D} \subseteq X$.

Key properties (Leiva 1991, Janfada 2010). For a C-quasi-semigroup with closed densely-defined generator $\{A(t)\}$:
- $x \in \mathcal{D} \Rightarrow K(t_0,s_0)x \in \mathcal{D}$ and $K(t_0,s_0)A(t)x = A(t)K(t_0,s_0)x$.
- $\frac{\partial}{\partial s}K(t,s)\,Cx_0 = A(t+s)K(t,s)Cx_0 = K(t,s)A(t+s)Cx_0$.
- Taking $r \to 0$ in (ii): $K(t,s)C = CK(t,s);\ K(t,s)x \in \mathrm{Rg}(C)$ for $x \in \mathrm{Rg}(C)$.

Canonical examples.
1. $K(t,s) = S(s)$ (C-semigroup) $\Rightarrow A(t) = A$.
2. $K(t,s) = S(g(t+s) - g(t)),\ g(t) = \int_0^t a(u)\,du,\ a > 0 \Rightarrow A(t) = a(t)A$.
3. $K(t,s) = Ce^{T(s+t)-T(t)}$ (C commutes with $T(t)$) $\Rightarrow A(t) = AT(t)$.

RESULTS & DISCUSSION

Hypotheses throughout: $\{K(t,s)\}_{t,s\geq 0}$ is a C-quasi-semigroup; $\{A(t)\}_{t\geq 0}$ closed, densely defined.

Lemmas

Lemma 4.1 (Bounded auxiliary operator). For $t,s \geq 0$ and $\lambda \in \mathbb{C}$, define

$$D_\lambda(t,s)\,x = \int_0^s e^{\lambda(s-h)}\,K(t-h,\,h)\,x\,dh.$$

Then $D_\lambda(t,s) \in \mathcal{B}(X)$.

Lemma 4.2 (Algebraic factorization). $\forall\,\lambda \in \mathbb{C}$, $t \geq s > 0$:
1. $\forall\,x \in \mathcal{D}$: $D_\lambda(t,s)\,(\lambda I - A(t))\,x = \big[e^{\lambda s}C - K(t-s,s)\big]x.$
2. $\forall\,x \in X$: $D_\lambda(t,s)\,x \in \mathcal{D}$ and $(\lambda I - A(t))\,D_\lambda(t,s)\,x = \big[e^{\lambda s}C - K(t-s,s)\big]x.$

Lemma 4.3 (Left decomposition). $\forall\,\lambda \in \mathbb{C}$, $t \geq s > 0$:

$$[\lambda I - A(t)]\;L_\lambda(t,s) + \varphi_\lambda(s)\,D_\lambda(t,s) = C,$$

where $L_\lambda(t,s) = \frac{1}{s}\int_0^s e^{-\lambda h} D_\lambda(t,h)\,dh$ and $\varphi_\lambda(s) = \frac{1}{s}e^{-\lambda s}$. Moreover, L_λ, D_λ, and $\lambda I - A(t)$ pairwise commute.

Corollaries

Corollary 4.1 (Kernel-Range inclusions). From Lemma 4.2, $\forall\,\lambda \in \mathbb{C}$, $n \in \mathbb{N}^*$, $t \geq s \geq 0$:
1. $\ker[\lambda I - A(t)] \subseteq \ker[e^{\lambda s}C - K(t-s,s)]$
2. $\mathrm{Rg}[e^{\lambda s}C - K(t-s,s)] \subseteq \mathrm{Rg}[\lambda I - A(t)]$
3. $\ker[\lambda I - A(t)]^n \subseteq \ker[e^{\lambda s}C - K(t-s,s)]^n$
4. $\mathrm{Rg}[e^{\lambda s}C - K(t-s,s)]^n \subseteq \mathrm{Rg}[\lambda I - A(t)]^n$
5. $\mathrm{Rg}^\infty[e^{\lambda s}C - K(t-s,s)] \subseteq \mathrm{Rg}^\infty[\lambda I - A(t)]$

Corollary 4.2 (Iterated factorizations). From Lemma 4.3, $\forall\,\lambda \in \mathbb{C}$, $n \in \mathbb{N}^*$, $t \geq s > 0$:
1. $\exists\,F_{\lambda,n}(t,s) \in \mathcal{B}(X)$ commuting with D_λ, L_λ: $[\lambda I - A(t)]^n[L_\lambda]^n + F_{\lambda,n}D_\lambda = C^n$
2. $\exists\,B_{\lambda,n}(t,s) \in \mathcal{B}(X)$ commuting with $D_\lambda, F_{\lambda,n}$: $[\lambda I - A(t)]^n B_{\lambda,n} + [F_{\lambda,n}]^n[D_\lambda]^n = C^{n^2}$

Proposition

Proposition 4.1 (Ascent & Descent bounds). $\forall\,n \in \mathbb{N}^*$:
1. $\mathrm{des}\big[e^{\lambda s}C - K(t-s,s)\big] = n \;\Rightarrow\; \mathrm{des}[\lambda I - A(t)] \leq n$
2. $\mathrm{asc}\big[e^{\lambda s}C - K(t-s,s)\big] = n \;\Rightarrow\; \mathrm{asc}[\lambda I - A(t)] \leq n$

where $\mathrm{asc}(T) = \inf\{k : \ker T^k = \ker T^{k+1}\}$, $\mathrm{des}(T) = \inf\{k : \mathrm{Rg}\,T^k = \mathrm{Rg}\,T^{k+1}\}$.

Main Theorems

Theorem 4.4 (Spectral inclusion — classical spectra). $\forall\,t \geq s \geq 0$:
1. $e^{s\,\sigma(A(t))} \subset \sigma(C, K(t-s,s)) \setminus \{0\}$
2. $e^{s\,\sigma_p(A(t))} \subset \sigma_p(C, K(t-s,s)) \setminus \{0\}$
3. $e^{s\,\sigma_{ap}(A(t))} \subset \sigma_{ap}(C, K(t-s,s)) \setminus \{0\}$
4. $e^{s\,\sigma_e(A(t))} \subset \sigma_e(C, K(t-s,s)) \setminus \{0\}$
5. $e^{s\,\sigma_r(A(t))} \subset \sigma_r(C, K(t-s,s)) \setminus \{0\}$

Theorem 4.5 (Ascent & Descent spectra). $\forall\,t \geq s > 0$:
1. $e^{s\,\sigma_{\mathrm{asc}}(A(t))} \subseteq \sigma_{\mathrm{asc}}(C, K(t-s,s)) \setminus \{0\}$
2. $e^{s\,\sigma_{\mathrm{des}}(A(t))} \subseteq \sigma_{\mathrm{des}}(C, K(t-s,s)) \setminus \{0\}$

Theorem 4.6 (Fine spectra: Saphar, Kato, Quasi-Fredholm). $\forall\,t \geq s > 0$:
- $e^{s\,\sigma_{\mathrm{Sap}}(A(t))} \subseteq \sigma_{\mathrm{Sap}}(C, K(t-s,s)) \setminus \{0\}$
- $e^{s\,\sigma_{e,\mathrm{Sap}}(A(t))} \subseteq \sigma_{e,\mathrm{Sap}}(C, K(t-s,s)) \setminus \{0\}$
- $e^{s\,\sigma_K(A(t))} \subseteq \sigma_K(C, K(t-s,s)) \setminus \{0\}$
- $e^{s\,\sigma_{eK}(A(t))} \subseteq \sigma_{eK}(C, K(t-s,s)) \setminus \{0\}$
- $e^{s\,\sigma_{qF}(A(t))} \subseteq \sigma_{qF}(C, K(t-s,s)) \setminus \{0\}$

CONCLUSION

We extended the spectral mapping theory from C_0-semigroups to the two-parameter setting of *regularized C-quasi-semigroups*. The master inclusion

$$\boxed{e^{s\,\sigma_\star(A(t))} \ \subseteq\ \sigma_\star(C,\, K(t-s,s)) \setminus \{0\}}$$

holds for all ten spectral types $\sigma_\star$ considered. The algebraic key is the pair of factorization identities (Lemmas 4.2 and 4.3), which encode the interplay between the generator family $\{A(t)\}$ and the quasi-semigroup $\{K(t,s)\}$. These results yield qualitative information on solutions of

$$x'(t) = A(t)x(t), \quad x(0) = Cx_0,$$

directly from spectral data of $\{A(t)\}_{t\geq 0}$.

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